Symmetric 3×3 matrices with repeated eigenvalues

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Abstract

Real symmetric 3×3 matrices have 6 independent entries (3 diagonal elements and 3 off-diagonal elements) and they have 3 real eigenvalues $(\lambda_0, \lambda_1, \lambda_2)$. If 2 of these 3 eigenvalues are the same, i. e., $\lambda_0 \neq \lambda_1 = \lambda_2$, then the number of *independent* entries of this matrix is reduced from 6 to 4. In this article, some relations between the matrix elements of such a symmetric 3×3 matrix with a repeated eigenvalue are presented, which reduce the number of degrees of freedom of the matrix from 6 to 4.

Contents

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1 Introduction

A general real symmetric matrix $\mathsf{S} = \mathsf{S}^{\mathrm{T}}$ is given by

$$\mathsf{S} = \begin{pmatrix} s_{00} & s_{01} & s_{02} \\ s_{01} & s_{11} & s_{12} \\ s_{02} & s_{12} & s_{22} \end{pmatrix}, \quad s_{mn} = s_{nm} \in \mathbb{R}. \tag{1}$$

Such a matrix has 6 independent entries: 3 diagonal elements (s_{00}, s_{11}, s_{22}) and 3 off-diagonal elements (s_{01}, s_{02}, s_{12}).

Each such matrix can be *diagonalized* [1, 2], i. e., there exists an orthogonal matrix R (with $\mathsf{R}^{-1} = \mathsf{R}^{\mathrm{T}}$) such that

$$\mathsf{R}^{\mathrm{T}}\,\mathsf{S}\,\mathsf{R} = \begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} = \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2), \quad \lambda_n \in \mathbb{R}. \tag{2}$$

The orthogonal matrix R contains the elements of the eigenvectors $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$ (written as column vectors) in its columns, i. e., $\mathsf{R} = \begin{bmatrix} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$. The eigenvectors and eigenvalues satisfy

$$\mathsf{S}\mathbf{v}_m = \lambda_m \mathbf{v}_m, \quad 0 \leq m < 3, \tag{3}$$

which is basically Eq. (2) multiplied (from the left) by R; Eq. (3) is then given columnwise by the resulting matrix equation.

The 3 eigenvectors $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$ build an orthonormal basis of $\mathbb{R}^3$ [1, 2], i. e., they have the scalar products $\mathbf{v}_m \cdot \mathbf{v}_n = \delta_{mn}$, where δ_{nm} is the Kronecker delta ($\delta_{mn} = 1$ for $m = n$ and $\delta_{mn} = 0$ for $m \neq n$). The latter property is of course equivalent to $\mathsf{R}^{\mathrm{T}}\,\mathsf{R} = \mathbb{1}$, which defines an orthogonal matrix.

To describe an orthonormal basis (i. e., the orthogonal matrix R), 3 parameters are required; different parametrizations exist such as:

- the 3 Euler angles (α, β, γ), which relate the orientation of an arbitrary orthonormal basis $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$ to the standard basis $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2)$ (where the n-th component $(\mathbf{e}_m)_n$ of $\mathbf{e}_m$ is $(\mathbf{e}_m)_n = \delta_{mn}$);
- two angles θ, ϕ (spherical coordinates) that define the orientation of $\mathbf{v}_0$ in space plus one more angle ξ that defines the orientation of $\mathbf{v}_1$ in the plane orthogonal to $\mathbf{v}_0$;
- any other parametrization of the rotation group in 3 dimensions (i. e., of the special[1] orthogonal group) SO(3), which is 3-dimensional itself (in general, SO(n) has dimension $n(n-1)/2$).

These 3 parameters defining R together with the 3 eigenvalues correspond to the 6 degrees of freedom of the original symmetric matrix S as defined in Eq. (1).

[1]The restriction to a determinant of 1 (i. e., to SO(3) instead of O(3)) corresponds to the choice of an appropriate order of the eigenvectors within R.

2 Repeated eigenvalues – results

If 2 of the 3 eigenvalues in Eq. (2) are the same, i. e., (without loss of generality) $\lambda_0 \neq \lambda_1 = \lambda_2$, then the symmetric matrix S can be written as (inverting Eq. (2)):

$$\mathsf{S} = \mathsf{R}\,\mathrm{diag}(\lambda_0, \lambda_1, \lambda_1)\,\mathsf{R}^{\mathrm{T}} = \mathsf{R}\begin{pmatrix}\lambda_0 & 0 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & \lambda_1\end{pmatrix}\mathsf{R}^{\mathrm{T}}. \tag{4}$$

Now, the 6 entries (cf. Eq. (1)) of the matrix S are no longer independent. Indeed, by setting $\lambda_1 = \lambda_2$, the number of parameters that are required to describe S is reduced from 6 to 4, since (a) only two eigenvalues are remaining and (b) the orientation of the eigenvector $\mathbf{v}_1$ (defined by ξ above) can now be chosen arbitrarily in the plane orthogonal to $\mathbf{v}_0$ (since, with $\mathbf{v}_1$ and $\mathbf{v}_2$, every linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ is also an eigenvector to the eigenvalue λ_1). Hence, 4 possible parameters describing S are λ_0, λ_1, and the two angles, θ, ϕ, describing the orientation of $\mathbf{v}_0$.

So, if S can be parametrized by only 4 parameters, then there must exist 2 (independent) equations relating the matrix elements $s_{00}, s_{11}, s_{22}, s_{01}, s_{02}, s_{12}$ to each other.

2.1 Main result

One such set of equations is

$$s_{00} = s_{22} + s_{02}\left(\frac{s_{01}}{s_{12}} - \frac{s_{12}}{s_{01}}\right) \tag{5}$$

$$s_{11} = s_{22} + s_{12}\left(\frac{s_{01}}{s_{02}} - \frac{s_{02}}{s_{01}}\right) \tag{6}$$

$$s_{00} = s_{11} + s_{01}\left(\frac{s_{02}}{s_{12}} - \frac{s_{12}}{s_{02}}\right). \tag{7}$$

Only 2 of these 3 equations are independent; e. g., the last equation can be found by subtracting the second one from the first one.

With any 2 of these 3 equations, 2 of the 3 diagonal elements (s_{00}, s_{11}, s_{22}) can be expressed by the 3 off-diagonal elements (s_{01}, s_{02}, s_{12}) plus the remaining third diagonal element (if the given off-diagonal elements are not zero).

2.2 Corollaries

Three other sets of equations, each of which expresses 2 elements (1 diagonal and 1 off-diagonal element) of S by the 4 other elements, can be derived from the results above by simple transformations. E. g., an expression for s_{02} can be easily obtained from Eq. (5), and a second equation for s_{11} can be derived by inserting this result for s_{02} into Eq. (6):

$$s_{02} = \frac{s_{01}s_{12}(s_{00} - s_{22})}{s_{01}^2 - s_{12}^2} \tag{8}$$

$$s_{11} = s_{22} + \frac{s_{01}^2 - s_{12}^2}{s_{00} - s_{22}} - s_{12}^2\frac{s_{00} - s_{22}}{s_{01}^2 - s_{12}^2} \tag{9}$$

(expressing s_{02} and s_{11} by $s_{00}, s_{22}, s_{01}, s_{12}$). Analog results derived from Eqs. (6) and (7) are

$$s_{12} = \frac{s_{01}s_{02}(s_{11} - s_{22})}{s_{01}^2 - s_{02}^2} \tag{10}$$

$$s_{00} = s_{11} + \frac{s_{01}^2 - s_{02}^2}{s_{11} - s_{22}} - s_{01}^2 \frac{s_{11} - s_{22}}{s_{01}^2 - s_{02}^2} \tag{11}$$

(expressing s_{12} and s_{00} by $s_{11}, s_{22}, s_{01}, s_{02}$). Finally, based on Eqs. (7) and (5), one finds

$$s_{01} = \frac{s_{02}s_{12}(s_{00} - s_{11})}{s_{02}^2 - s_{12}^2} \tag{12}$$

$$s_{22} = s_{00} + \frac{s_{02}^2 - s_{12}^2}{s_{00} - s_{11}} - s_{02}^2 \frac{s_{00} - s_{11}}{s_{02}^2 - s_{12}^2} \tag{13}$$

(expressing s_{01} and s_{22} by $s_{00}, s_{11}, s_{02}, s_{12}$).

Again, the results require that certain sub-expressions are not zero such as $s_{01}^2 - s_{12}^2$ and $s_{00} - s_{22}$ in the first set of equations.

2.3 More corollaries

Some relations for the off-diagonal entries of S can be expressed as quadratic equations, which can be easily solved; these results are found by rearranging the first equations (Eqs. (8), (10), (12)) of the previous subsection:

$$s_{12}^2 + \frac{s_{01}}{s_{02}}(s_{00} - s_{22})\, s_{12} - s_{01}^2 = 0, \tag{14}$$

$$s_{01}^2 + \frac{s_{02}}{s_{12}}(s_{22} - s_{11})\, s_{01} - s_{02}^2 = 0, \tag{15}$$

$$s_{02}^2 + \frac{s_{12}}{s_{01}}(s_{11} - s_{00})\, s_{02} - s_{12}^2 = 0. \tag{16}$$

Finally, quadratic equations for the *squared* off-diagonal elements depending on all 3 diagonal elements(s_{00}, s_{11}, s_{22}) and only 1 other off-diagonal element can be found by combining 2 of the 3 Eqs. (8), (10), and (12). Inserting Eq. (10) into Eq. (8) yields for s_{02}^2:

$$(s_{02}^2)^2 + \Big((s_{00} - s_{11})(s_{11} - s_{22}) - 2s_{01}^2\Big)s_{02}^2 + s_{01}^2\Big(s_{01}^2 - (s_{00} - s_{22})(s_{11} - s_{22})\Big) = 0 \tag{17}$$

and inserting Eq. (8) into Eq. (10) yields for s_{12}^2:

$$(s_{12}^2)^2 + \Big((s_{00} - s_{11})(s_{22} - s_{00}) - 2s_{01}^2\Big)s_{12}^2 + s_{01}^2\Big(s_{01}^2 - (s_{00} - s_{22})(s_{11} - s_{22})\Big) = 0, \tag{18}$$

so, by solving these two quadratic equations, the off-diagonal elements s_{02} and s_{12} can obtained from the 3 diagonal elements and the remaining off-diagonal element s_{01}.

Similarly, inserting first Eq. (12) into Eq. (8) and then Eq. (10) into Eq. (12) yields:

$$(s_{12}^2)^2 + \Big((s_{00} - s_{22})(s_{11} - s_{00}) - 2s_{02}^2\Big)s_{12}^2 + s_{02}^2\Big(s_{02}^2 + (s_{00} - s_{11})(s_{11} - s_{22})\Big) = 0, \quad (19)$$

$$(s_{01}^2)^2 + \Big((s_{11} - s_{22})(s_{22} - s_{00}) - 2s_{02}^2\Big)s_{01}^2 + s_{02}^2\Big(s_{02}^2 + (s_{00} - s_{11})(s_{11} - s_{22})\Big) = 0; \quad (20)$$

and inserting first Eq. (12) into Eq. (8) and then Eq. (8) into Eq. (12) yields:

$$(s_{01}^2)^2 + \Big((s_{00} - s_{22})(s_{22} - s_{11}) - 2s_{12}^2\Big)s_{01}^2 + s_{12}^2\Big(s_{12}^2 - (s_{00} - s_{11})(s_{00} - s_{22})\Big) = 0, \quad (21)$$

$$(s_{02}^2)^2 + \Big((s_{00} - s_{11})(s_{11} - s_{22}) - 2s_{12}^2\Big)s_{02}^2 + s_{12}^2\Big(s_{12}^2 - (s_{00} - s_{11})(s_{00} - s_{22})\Big) = 0. \quad (22)$$

2.4 Summarized results

The following table summarizes which of the previous results are relevant in different situations depending on the given and dependent entries in the symmetric matrix S. (Obviously, not all possible combinations with two given diagonal elements are included.)

Table 1: Summary of results

given elements		dependent elements	required equations
diagonal	off-diagonal		
s_{00}	s_{01}, s_{02}, s_{12}	s_{11}, s_{22}	(5), (7)
s_{11}	s_{01}, s_{02}, s_{12}	s_{00}, s_{22}	(6), (7)
s_{22}	s_{01}, s_{02}, s_{12}	s_{00}, s_{11}	(5), (6)
s_{00}, s_{11}	s_{02}, s_{12}	s_{22}, s_{01}	(12), (13)
s_{00}, s_{22}	s_{01}, s_{12}	s_{11}, s_{02}	(8), (9)
s_{11}, s_{22}	s_{01}, s_{02}	s_{00}, s_{12}	(10), (11)
s_{00}, s_{11}, s_{22}	s_{01}	s_{02}, s_{12}	(17), (18)
s_{00}, s_{11}, s_{22}	s_{02}	s_{01}, s_{12}	(19), (20)
s_{00}, s_{11}, s_{22}	s_{12}	s_{01}, s_{02}	(21), (22)

3 How to check the result

3.1 Parametrizing the matrix

The diagonalized form of a symmetric 3×3 matrix with a repeated eigenvalue can be written as

$$\begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} = \lambda_1 \mathbb{1} + (\lambda_0 - \lambda_1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{23}$$

The general (i. e., not diagonalized) form of such a symmetric matrix can be found by multiplying this expression by R and R^{T} as in Eq. (4):

$$\mathsf{S} = \lambda_1 \, \mathsf{R} \, \mathbb{1} \, \mathsf{R}^{\mathrm{T}} + (\lambda_0 - \lambda_1) \, \mathsf{R} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathsf{R}^{\mathrm{T}}. \tag{24}$$

With $\mathsf{R} = [\mathbf{v}_0 \; \mathbf{v}_1 \; \mathbf{v}_2]$, one finds

$$\mathsf{R} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathsf{R}^{\mathrm{T}} = \begin{bmatrix} \mathbf{v}_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_0^{\mathrm{T}} \\ \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{bmatrix} = \mathbf{v}_0 \mathbf{v}_0^{\mathrm{T}} = \mathbf{v}_0 \otimes \mathbf{v}_0 \tag{25}$$

where $\mathbf{v}_0 \otimes \mathbf{v}_0$ is the dyadic (or outer) product of the vector $\mathbf{v}_0$ with itself. Using $\mathsf{R} \, \mathbb{1} \, \mathsf{R}^{\mathrm{T}} = \mathbb{1}$ and combining the last two equations yields

$$\mathsf{S} = \lambda_1 \, \mathbb{1} + (\lambda_0 - \lambda_1)(\mathbf{v}_0 \otimes \mathbf{v}_0). \tag{26}$$

Writing $\mathbf{v}_0 = (r, s, t)^{\mathrm{T}}$, the dyadic product is

$$\mathbf{v}_0 \otimes \mathbf{v}_0 = \begin{pmatrix} rr & rs & rt \\ rs & ss & st \\ rt & st & tt \end{pmatrix} \tag{27}$$

and since $\mathbf{v}_0$ is normalized to length 1, i. e., $\mathbf{v}_0 \cdot \mathbf{v}_0 = r^2 + s^2 + t^2 = 1$, one can substitute $t = \pm\sqrt{1 - r^2 - s^2}$ to obtain:

$$\mathsf{S} = \lambda_1 \, \mathbb{1} + (\lambda_0 - \lambda_1) \begin{pmatrix} r^2 & rs & \pm r\sqrt{1 - r^2 - s^2} \\ rs & s^2 & \pm s\sqrt{1 - r^2 - s^2} \\ \pm r\sqrt{1 - r^2 - s^2} & \pm s\sqrt{1 - r^2 - s^2} & 1 - r^2 - s^2 \end{pmatrix} \tag{28}$$

or, combined[2]:

$$\mathsf{S} = \begin{pmatrix} \lambda_1 + (\lambda_0 - \lambda_1) r^2 & (\lambda_0 - \lambda_1) rs & \pm(\lambda_0 - \lambda_1) r\sqrt{1 - r^2 - s^2} \\ (\lambda_0 - \lambda_1) rs & \lambda_1 + (\lambda_0 - \lambda_1) s^2 & \pm(\lambda_0 - \lambda_1) s\sqrt{1 - r^2 - s^2} \\ \pm(\lambda_0 - \lambda_1) r\sqrt{1 - r^2 - s^2} & \pm(\lambda_0 - \lambda_1) s\sqrt{1 - r^2 - s^2} & \lambda_1 + (\lambda_0 - \lambda_1)(1 - r^2 - s^2) \end{pmatrix} \tag{29}$$

[2] Some helpful relations, which can be obtained from Eq. (29), are $\frac{r}{s} = \frac{s_{02}}{s_{12}}$ as well as $\lambda_1 = s_{00} - s_{01} \frac{s_{02}}{s_{12}}$, $\lambda_1 = s_{11} - s_{01} \frac{s_{12}}{s_{02}}$, and $\lambda_1 = s_{22} - \frac{s_{02} s_{12}}{s_{01}}$.

3.2 Putting things together

The main result (Eqs. (5), (6), (7)) can now be checked by substituting the expressions from Eq. (29). For Eq. (5), one finds (starting with the right-hand side of the equation):

$$\begin{aligned}
&s_{22} + s_{02}\left(\frac{s_{01}}{s_{12}} - \frac{s_{12}}{s_{01}}\right) \\
&= s_{22} + s_{02}\frac{s_{01}^2 - s_{12}^2}{s_{12}s_{01}} \\
&= s_{22} + s_{02}\,\frac{(\lambda_0-\lambda_1)^2 r^2 s^2 - (\lambda_0-\lambda_1)^2 s^2(1-r^2-s^2)}{\pm(\lambda_0-\lambda_1)s\sqrt{1-r^2-s^2}(\lambda_0-\lambda_1)rs} \\
&= s_{22} + s_{02}\,\frac{2r^2-1+s^2}{\pm r\sqrt{1-r^2-s^2}} \\
&= s_{22} + \left((\lambda_0-\lambda_1)r\sqrt{1-r^2-s^2}\right)\frac{2r^2-1+s^2}{r\sqrt{1-r^2-s^2}} \\
&= s_{22} + (\lambda_0-\lambda_1)(2r^2-1+s^2) \\
&= \left(\lambda_1 + (\lambda_0-\lambda_1)(1-r^2-s^2)\right) + (\lambda_0-\lambda_1)(2r^2-1+s^2) \\
&= \lambda_1 + (\lambda_0-\lambda_1)r^2 \\
&= s_{00}.
\end{aligned} \tag{30}$$

Similarly, for Eq. (6):

$$\begin{aligned}
&s_{22} + s_{12}\left(\frac{s_{01}}{s_{02}} - \frac{s_{02}}{s_{01}}\right) \\
&= s_{22} + s_{12}\frac{s_{01}^2 - s_{02}^2}{s_{02}s_{01}} \\
&= s_{22} + s_{12}\,\frac{(\lambda_0-\lambda_1)^2 r^2 s^2 - (\lambda_0-\lambda_1)^2 r^2(1-r^2-s^2)}{\pm(\lambda_0-\lambda_1)r\sqrt{1-r^2-s^2}(\lambda_0-\lambda_1)rs} \\
&= s_{22} + s_{12}\,\frac{2s^2-1+r^2}{\pm s\sqrt{1-r^2-s^2}} \\
&= s_{22} + \left((\lambda_0-\lambda_1)s\sqrt{1-r^2-s^2}\right)\frac{2s^2-1+r^2}{s\sqrt{1-r^2-s^2}} \\
&= s_{22} + (\lambda_0-\lambda_1)(2s^2-1+r^2) \\
&= \left(\lambda_1 + (\lambda_0-\lambda_1)(1-r^2-s^2)\right) + (\lambda_0-\lambda_1)(2s^2-1+r^2) \\
&= \lambda_1 + (\lambda_0-\lambda_1)s^2 \\
&= s_{11}.
\end{aligned} \tag{31}$$

As mentioned above, Eq. (7) can be obtained by subtracting Eq. (6) from Eq. (5).

References

[1] Anton H. Elementary Linear Algebra. 10th ed. John Wiley & Sons, Hoboken 2010.

[2] Datta KB. Matrix and Linear Algebra. Prentice-Hall of India, New Delhi 1991.

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Thanks to Orodruin at physicsforms.com, who suggested to use Eq. (23).

Manuscript version history

2016-05-19 first version

2016-05-20 update:

- minor corrections in the introduction
- removed trivial solutions to quadratic equations
- added more quadratic equations to calculate 2 off-diagonal elements if all 3 diagonal elements are given

2016-06-02 update:

- re-arranged Eqs. (30) and (31) to clarify the reasoning
- added auxiliary relations in footnote 2
- added Table 1