Diffusion between permeable membranes: Calculation of the initial slope of the time-dependent diffusion coefficient

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Abstract

Molecular diffusion in a (one-dimensional) system of equidistant permeable membranes results in a time-dependent (effective) diffusion coefficient $D_{\text{eff}}(\tau)$, which decreases with increasing diffusion time τ. In this article, the time dependence of $D_{\text{eff}}(\tau)$ for very short diffusion times τ is calculated as a function of the membrane distance L and the permeability P.

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1. Introduction

Diffusion (i. e., the microscopical thermal random motion) of water molecules in biological structures and tissues is an important phenomenon, which can be exploited to measure microstructural properties. Mathematically, molecular diffusion is quantitatively characterized by a probability distribution $p(\Delta x, \tau;\, D_0, P)$ for a certain molecular displacement Δx (the diffusion distance) during a distribution time (interval) τ; the probability distribution depends on the diffusion coefficient D_0 (the free diffusion coefficient of the substance without any diffusion restrictions) and – possibly – on other parameters P.

For *unrestricted* (or *free*) one-dimensional diffusion of the molecules of a pure liquid, this probability distribution is a Gaussian distribution with variance $\sigma^2 = 2D_0\tau$. Consequently, the average diffusion distance of a particle is proportional to the square root of τ. In general, the probability distribution has a more complicated form and depends on more parameters describing, e. g., the spacing and permeability of membranes in biological tissues. If diffusion is restricted by such membranes, the mean displacement is smaller than in the case of free diffusion and the measured effective diffusion coefficient decreases with increasing diffusion time in a characteristic way depending on the confining geometry. The time-dependent diffusion coefficient $D(\tau)$ is here defined as the constant of proportionality between the mean square displacement $\langle(\Delta x)^2\rangle$ of the diffusing molecules and the diffusion time τ ([Sen, 2004, Einstein, 1956, Jensen and Helpern, 2010, Grebenkov, 2007]):

$$\langle(\Delta x)^2\rangle = E((\Delta x)^2) = 2dD(\tau)\tau \tag{1}$$

where $E(\cdot)$ denotes the expectation value and d is the dimensionality of the system (in the following, we will only consider one-dimensional systems, i. e., $d = 1$).

This time dependence of $D(\tau)$ can be used to determine the local microstructural properties (such as membrane distance and permeability) in biological tissue by measuring the effective diffusion coefficient at different diffusion times and fitting the data to a model of restricted diffusion.

In this article, the probability distribution of molecular diffusion in a one-dimensional system of equidistant permeable membranes is analyzed with respect to the dependence of the effective diffusion coefficient on the diffusion time, the membrane distance, and the membrane permeability.

2. Diffusion between permeable barriers

The diffusion model under consideration is a one-dimensional model of equidistant barriers with distance L and permeability P; the permeability is here defined in units of an inverse length (m^{-1})[1]

[1] A frequently found alternative definition of the membrane permeability in a system with molecular diffusion described by free diffusion coefficient D_0 is $P' = P \cdot D_0$, i. e., in units of m/s.

2.1. The original model

The probability distribution p of the diffusion displacement in an infinite one-dimensional array of permeable barriers was derived by Powles et al. [1992] as:

$$
\begin{aligned}
\sqrt{4\pi D_0\tau}\, p(x,\tau;\, x_0;\, D_0, L, P) \;=\;& (1-f)^{|z|} \exp\left(-(|x-x_0|)^2/4D_0\tau\right) \\
&+ \sum_{\substack{l=|z|\\ \text{step } 2}}^{\infty} (1-f)^l f \left(\exp\left(\frac{-\big((l+1-z)L+x+x_0\big)^2}{4D_0\tau} \right) \right. \\
&\qquad \left. + \exp\left(\frac{-\big((l+1+z)L-x-x_0\big)^2}{4D_0\tau} \right) \right) \\
&+ \sum_{\substack{l=|z|\\ \text{step } 2}}^{\infty} \sum_{k=2}^{\infty} (1-f)^l f^k \prod_{j=2}^{k} \left(\frac{2j-1-(2z-1)(-1)^j+2l}{2j-1+(-1)^j} \right) \\
&\qquad \cdot \exp\left(\frac{-\big((l+k+(-1)^k z)L-(-1)^k x+x_0\big)^2}{4D_0\tau} \right) \\
&+ \sum_{\substack{l=|z|\\ \text{step } 2}}^{\infty} \sum_{k=2}^{\infty} (1-f)^l f^k \prod_{j=2}^{k} \left(\frac{2j-1+(2z+1)(-1)^j+2l}{2j-1+(-1)^j} \right) \\
&\qquad \cdot \exp\left(\frac{-\big((l+k-(-1)^k z)L+(-1)^k x-x_0\big)^2}{4D_0\tau} \right)
\end{aligned}
$$

with barrier distance L, permeability P, free diffusivity D_0, initial particle position x_0; z describes the 'slice number' of the position x, i. e., $(z-\frac{1}{2})L < x < (z+\frac{1}{2})L$, and k and l give the number of reflections and transmissions, respectively. $f = f(\mathcal{X};\tau,P,D_0)$ is the reflection coefficient with

$$
f(\mathcal{X};\tau,P,D_0) = 1 - 2P \int_0^{\infty} \mathrm{d}s\, \exp\left(-\left(2Ps + \frac{s^2}{4D_0\tau} + \frac{s}{2D_0\tau}\mathcal{X} \right) \right),
$$

where $\mathcal{X}$ is to be substituted with the spatial argument of the exponential function multiplied with f.

2.2. A dimensionless model

As recently shown [Dietrich et al., 2014], this probability distribution can be simplified by introducing dimensionless quantities proportional to x, x_0, τ, P defining $\tilde{x} = x/L, \tilde{x}_0 = x_0/L$, $\tilde{\tau} = 2D_0\tau/L^2$ (i. e., $\tilde{\tau}$ is the variance of the original, unrestricted diffusion distribution), and $\tilde{P} = LP$. With these new quantities, the probability density p (having the units of an inverse length) is replaced by the dimensionless density $\tilde{p} = Lp$ (with normalization $\int_{-\infty}^{\infty} \tilde{p}(\tilde{x})\mathrm{d}\tilde{x} = 1$, cf. Fig. 1):

$$\begin{aligned}
\sqrt{2\pi\tilde{\tau}}\;\tilde{p}(\tilde{x},\tilde{\tau};\,\tilde{x}_0;\,\tilde{P}) \;=\;& (1-f)^{|z|}\exp\big(-(|\tilde{x}-\tilde{x}_0|)^2/2\tilde{\tau}\big)\\
&+ \sum_{\substack{l=|z|\\ \text{step } 2}}^{\infty} (1-f)^l f \;\left(\exp\left(\frac{-\big((l+1-z)+\tilde{x}+\tilde{x}_0\big)^2}{2\tilde{\tau}}\right)\right.\\
&\qquad\qquad\left. + \exp\left(\frac{-\big((l+1+z)-\tilde{x}-\tilde{x}_0\big)^2}{2\tilde{\tau}}\right)\right)\\
&+ \sum_{\substack{l=|z|\\ \text{step } 2}}^{\infty}\sum_{k=2}^{\infty}(1-f)^l f^k \prod_{j=2}^{k}\left(\frac{2j-1-(2z-1)(-1)^j+2l}{2j-1+(-1)^j}\right)\\
&\qquad\qquad \cdot\exp\left(\frac{-\big((l+k+(-1)^k z)-(-1)^k\tilde{x}+\tilde{x}_0\big)^2}{2\tilde{\tau}}\right)\\
&+ \sum_{\substack{l=|z|\\ \text{step } 2}}^{\infty}\sum_{k=2}^{\infty}(1-f)^l f^k \prod_{j=2}^{k}\left(\frac{2j-1+(2z+1)(-1)^j+2l}{2j-1+(-1)^j}\right)\\
&\qquad\qquad \cdot\exp\left(\frac{-\big((l+k-(-1)^k z)+(-1)^k\tilde{x}-\tilde{x}_0\big)^2}{2\tilde{\tau}}\right)
\end{aligned}$$

and the reflection coefficient $f = f(\tilde{\mathcal{X}};\tilde{\tau},\tilde{P})$ is then calculated after replacing s by $\tilde{s} = s/L$ as

$$f(\tilde{\mathcal{X}};\tilde{\tau},\tilde{P}) = 1 - 2\tilde{P}\int_0^{\infty} \mathrm{d}\tilde{s}\;\exp\left(-\left(2\tilde{P}\tilde{s} + \frac{\tilde{s}^2}{2\tilde{\tau}} + \frac{\tilde{s}}{\tilde{\tau}}\tilde{\mathcal{X}}\right)\right),$$

where $\tilde{\mathcal{X}}$ is again to be substituted with the dimensionless ("spatial") argument of the exponential function multiplied with f. This integral can be solved using the error function $\mathrm{erf}(x)$ as well as the complementary error function $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$ and is[2] [3]

$$\begin{aligned}
f(\tilde{\mathcal{X}};\,\tilde{\tau},\tilde{P}) &= 1 - 2\tilde{P}\int_0^{\infty}\mathrm{d}\tilde{s}\;\exp-\left(\left(2\tilde{P}+\frac{\tilde{\mathcal{X}}}{\tilde{\tau}}\right)\tilde{s} + \frac{1}{2\tilde{\tau}}\tilde{s}^2\right)\\
&= 1 - 2\tilde{P}\left[\sqrt{\frac{\pi}{2}}\sqrt{\tilde{\tau}}\;\exp\left(\frac{\tilde{\tau}}{2}\Big(2\tilde{P}+\frac{\tilde{\mathcal{X}}}{\tilde{\tau}}\Big)^2\right)\mathrm{erf}\left(\frac{s}{\sqrt{2\tilde{\tau}}} + \sqrt{\frac{\tilde{\tau}}{2}}\Big(2\tilde{P}+\frac{\tilde{\mathcal{X}}}{\tilde{\tau}}\Big)\right)\right]_0^{\infty}\\
&= 1 - 2\tilde{P}\sqrt{\frac{\pi}{2}}\sqrt{\tilde{\tau}}\;\exp\left(\frac{\tilde{\tau}}{2}\Big(2\tilde{P}+\frac{\tilde{\mathcal{X}}}{\tilde{\tau}}\Big)^2\right)\mathrm{erfc}\left(\sqrt{\frac{\tilde{\tau}}{2}}\Big(2\tilde{P}+\frac{\tilde{\mathcal{X}}}{\tilde{\tau}}\Big)\right).
\end{aligned}$$

[2]The integral of $\exp(-(ax+bx^2))$ is $\frac{\sqrt{\pi}}{2\sqrt{b}}\exp(\frac{a^2}{4b})\mathrm{erf}(\sqrt{b}x+\frac{a}{2\sqrt{b}})$.

[3]Unfortunately, the product $\exp(x^2)\mathrm{erfc}(x)$ is somewhat difficult to evaluate numerically (since the first factor goes to infinity rather quickly and the second one to zero). However, the product can be approximated e. g. as $\left[\frac{1}{2}\sqrt{\pi}\left(x+\sqrt{2[1-(1-2/\pi)\exp(-x\sqrt{5/7})]+x^2}\right)\right]^{-1}$ according to a result by Spanier & Oldham given by Lether [1990, Eq. (2)]; see also Kim et al. [1990].

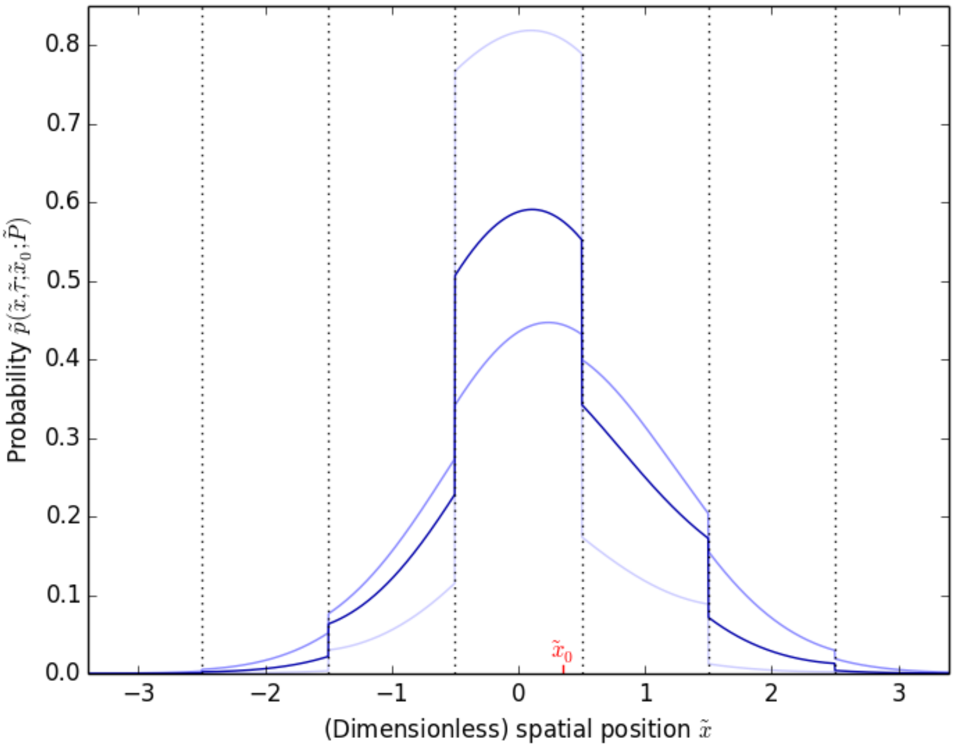

Figure 1: Probability distributions for diffusion through permeable membranes (dotted vertical lines) with permeabilities of $\tilde{P} = 1.0$ (dark blue), $\tilde{P} = 4.0$ (medium blue), and $\tilde{P} = 0.25$ (light blue). The (dimensionless) diffusion time was set to $\tilde{\tau} = 1.0$ and the initial position was $\tilde{x}_0 = 0.35$.

2.3. Calculation of the effective diffusion coefficient

Integrating $\tilde{p}$ over all initial positions $-\frac{1}{2} < \tilde{x}_0 < \frac{1}{2}$ gives the averaged probability density $\bar{p}$ for a displacement $\Delta\tilde{x} = \tilde{x} - \tilde{x}_0$:

$$\bar{p}(\Delta\tilde{x};\, \tilde{\tau}, \tilde{P}) = \int_{-1/2}^{1/2} \mathrm{d}\tilde{x}_0\, \tilde{p}(\Delta\tilde{x} + \tilde{x}_0, \tilde{\tau};\, \tilde{x}_0;\, \tilde{P}).$$

The dimensionless time-dependent diffusion coefficient $\tilde{D}_{\text{eff}} = D_{\text{eff}}/(2D_0)$ is calculated according to Eq. 1 as

$$\tilde{D}_{\text{eff}}(\tilde{\tau};\, \tilde{P}) = \frac{1}{2\tilde{\tau}} E\big((\Delta\tilde{x})^2\big).$$

In the case of purely Gaussian diffusion, the dimensionless diffusion coefficient is (*per* *definitionem*) simply $\frac{1}{2}$ and can be obtained from the variance as

$$\tilde{D}_{\text{eff}}(\tilde{\tau};\, \tilde{P}) = \frac{1}{2\tilde{\tau}} E\big((\Delta\tilde{x})^2\big) = \frac{1}{2}.$$

3. Summary of results

The main result derived in the following sections is the form of $\tilde{D}_{\text{eff}}(\tilde{\tau})$ for very small values of $\tilde{\tau}$. This result is first derived in the simpler case of impermeable membranes, i. e., for $\tilde{P} = 0$ and then for the more general case $\tilde{P} > 0$.

In the case of impermeable membranes, the final result for small values $\tilde{\tau} \ll 1$ is (cf. Eq. A.1)

$$\tilde{D}_{\text{eff}}(\sqrt{\tilde{\tau}}) \approx \frac{1}{\tilde{\tau}}\Big(\frac{\tilde{\tau}}{2} - \frac{2}{3}\sqrt{\frac{2}{\pi}}\sqrt{\tilde{\tau}}^3\Big) = \frac{1}{2} - \frac{2}{3}\sqrt{\frac{2}{\pi}}\sqrt{\tilde{\tau}} \approx 0.5 - 0.531\,923\sqrt{\tilde{\tau}}. \tag{2}$$

If this is translated back to physical units using $\tilde{D}_{\text{eff}} = D_{\text{eff}}/(2D_0)$ and $\tilde{\tau} = 2D_0\tau/L^2$, the result reads

$$D_{\text{eff}}(\sqrt{\tau}) \approx 2D_0\Big(\frac{1}{2} - \frac{2}{3}\sqrt{\frac{2}{\pi}}\frac{\sqrt{2D_0\tau}}{L}\Big) = D_0\Big(1 - \frac{8}{3\sqrt{\pi}L}\sqrt{D_0\tau}\Big), \tag{3}$$

which is exactly the well-known result by Mitra et al. [1992].

The more interesting result is the influence of permeable membranes (with permeability $\tilde{P} > 0$) on the effective diffusion coefficient at small diffusion times $\tilde{\tau} \ll 1$, which is approximately (cf. Eq. B.5):

$$\tilde{D}_{\text{eff}}(\sqrt{\tilde{\tau}}) \approx \frac{1}{2} - \frac{1}{2\tilde{P}} - \frac{1}{4\tilde{\tau}\tilde{P}^3} + \frac{1}{\tilde{P}^2\sqrt{2\pi\tilde{\tau}}} + \frac{1}{4\tilde{\tau}\tilde{P}^3}\exp(2\tilde{\tau}\tilde{P}^2)\operatorname{erfc}(\sqrt{2\tilde{\tau}}\tilde{P}). \tag{4}$$

Expressing the last term of this result as a power series in $\sqrt{\tilde{\tau}}$ yields the final expression (cf. Eq. B.6):

$$\begin{aligned}\tilde{D}_{\text{eff}}(\sqrt{\tilde{\tau}}) \approx \frac{1}{2} - \frac{2}{3}\sqrt{\frac{2}{\pi}}\sqrt{\tilde{\tau}} + \frac{\tilde{P}}{2}\tilde{\tau} - \frac{8}{15}\sqrt{\frac{2}{\pi}}\tilde{P}^2\,\tilde{\tau}^{3/2} + \frac{\tilde{P}^3}{3}\tilde{\tau}^2 \\ - \frac{32}{105}\sqrt{\frac{2}{\pi}}\tilde{P}^4\,\tilde{\tau}^{5/2} + \frac{\tilde{P}^5}{6}\tilde{\tau}^3 - + \cdots\end{aligned} \tag{5}$$

(The results of Eq. 4 and Eq. 5 are illustrated in Fig. 2.)

The important point here is that the first derivative of $\tilde{D}_{\text{eff}}$ with respect to $\tilde{\sigma} = \sqrt{\tilde{\tau}}$ is the same as for impermeable membranes (cf. Eq. (2)) and only the second (and higher) derivatives do depend on $\tilde{P}$. In particular, we have the second derivative (in lowest order)

$$\frac{\mathrm{d}^2\tilde{D}_{\text{eff}}}{\mathrm{d}\tilde{\sigma}^2} \approx \tilde{P}. \tag{6}$$

Translating this result back to physical units (using $\tilde{P} = LP$) gives (to second order in $\sqrt{\tau}$):

$$D_{\text{eff}}(\sqrt{\tau}) \approx D_0\Big(1 - \frac{8}{3\sqrt{\pi}L}\sqrt{D_0\tau} + \frac{2P}{L}D_0\tau\Big). \tag{7}$$

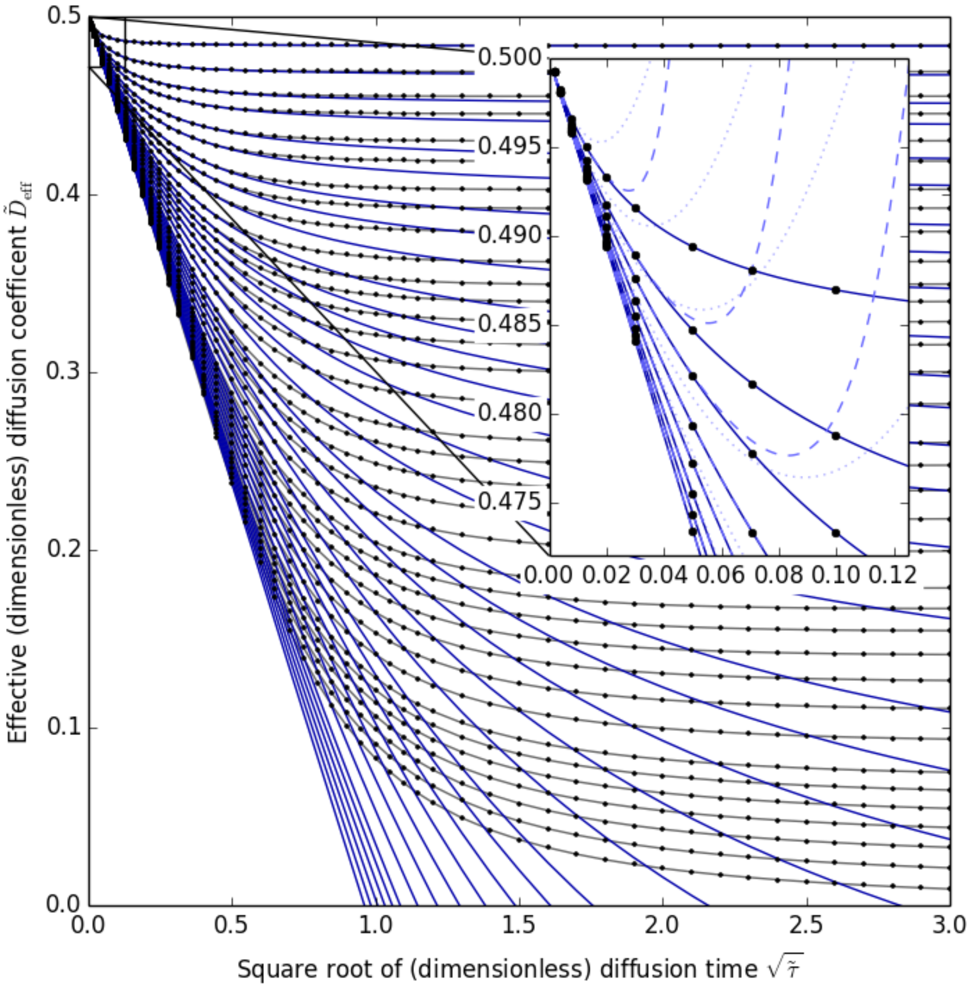

Figure 2: Simulation results and approximations of the dimensionless diffusion coefficient for different permeabilities. Based on the dimensionless model introduced in section 2.2, the effective diffusion coefficient was determined numerically (black circles, connected with gray lines) for permeabilities $\tilde{P}$ in the range from 0.025 (bottom curve) to 30.0 (top curve). The approximation given in Eq. 4 based on the product $\exp(2\tilde{\tau}\tilde{P}^2)\text{erfc}(\sqrt{2\tilde{\tau}}\tilde{P})$ is shown as solid blue line and agrees very well with the simulations for short diffusion times. The power-series approximation given in Eq. 5 is shown in the inset magnification (for a selection of 8 permeabilities) to order $(\sqrt{\tilde{\tau}})^2$ (dotted) and to order $(\sqrt{\tilde{\tau}})^6$ (dashed). These approximations agree with the simulations for sufficiently small values of $\tilde{\tau}$.

Appendices

Appendix A. Calculation 1: Initial slope for $\tilde{P} = 0$

First, we calculate an approximation of $\tilde{p}(\Delta\tilde{x}, \tilde{\tau}; \tilde{P})$ for $\tilde{P} = 0$ (impermeable membranes) and $\tilde{\tau} \to 0$ (very short diffusion times) in order to estimate the initial slope of $\tilde{D}_{\text{eff}}(\sqrt{\tilde{\tau}}; \tilde{P})$:

For very short $\tilde{\tau} \to 0$, the displacement of a single particle starting at $\tilde{x}_0$ is a very narrow Gaussian $g_{\tilde{\tau}}(\tilde{x} - \tilde{x}_0) = (\sqrt{2\pi\tilde{\tau}})^{-1} \exp\big(-(\tilde{x} - \tilde{x}_0)^2/(2\tilde{\tau})\big)$ with standard deviation $\sqrt{\tilde{\tau}}$ for most initial points (and only considerably different from Gaussian, if $\tilde{x}_0$ is close to $\pm\frac{1}{2}$, i. e., close to a barrier). For $\tilde{P} = 0$, i. e., for impermeable membranes, the behavior of diffusion can be described by a single reflection at either the left or the right barrier (only one reflection needs to be considered since the standard deviation of the Gaussian is very small ($\ll 1$) at short $\tilde{\tau}$. Hence, the probability distribution for a particle with starting point $-\frac{1}{2} < \tilde{x}_0 < \frac{1}{2}$ is approximately

$$\tilde{p}(\tilde{x}, \tilde{x}_0) \approx \Big(g_{\tilde{\tau}}(\tilde{x} - \tilde{x}_0) + g_{\tilde{\tau}}\big(\tilde{x} - (1 - \tilde{x}_0)\big) + g_{\tilde{\tau}}\big(\tilde{x} - (-1 - \tilde{x}_0)\big)\Big) B(\tilde{x})$$

where $B(\tilde{x})$ is a box function, which is 1 for $-\frac{1}{2} < \tilde{x} < \frac{1}{2}$ and 0 elsewhere.

The displacement probability $\bar{p}(\Delta\tilde{x})$ for $\Delta\tilde{x} = \tilde{x} - \tilde{x}_0$ is obtained by integration over all initial positions $-\frac{1}{2} < \tilde{x}_0 < \frac{1}{2}$, i. e.

$$\begin{aligned}
\bar{p}(\Delta\tilde{x}) &\approx \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big(g_{\tilde{\tau}}(\Delta\tilde{x}) + g_{\tilde{\tau}}\big(\Delta\tilde{x} - (1 - 2\tilde{x}_0)\big)\big) + g_{\tilde{\tau}}\big(\Delta\tilde{x} - (-1 - 2\tilde{x}_0)\big)\Big) B(\Delta\tilde{x} + \tilde{x}_0)\, \mathrm{d}\tilde{x}_0 \\
&= \underbrace{g_{\tilde{\tau}}(\Delta\tilde{x}) \int_{-\frac{1}{2}}^{\frac{1}{2}} B(\Delta\tilde{x} + \tilde{x}_0)\, \mathrm{d}\tilde{x}_0}_{=\bar{p}_1(\Delta\tilde{x})} + \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} g_{\tilde{\tau}}\big(\Delta\tilde{x} - (1 - 2\tilde{x}_0)\big)\big) B(\Delta\tilde{x} + \tilde{x}_0)\, \mathrm{d}\tilde{x}_0}_{=\bar{p}_2(\Delta\tilde{x})} \\
&\quad + \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} g_{\tilde{\tau}}\big(\Delta\tilde{x} - (-1 - 2\tilde{x}_0)\big)\big) B(\Delta\tilde{x} + \tilde{x}_0)\, \mathrm{d}\tilde{x}_0}_{=\bar{p}_3(\Delta\tilde{x})}
\end{aligned}$$

This integral can be split into 3 parts: the first one is the non-reflected diffusion distribution

$$\bar{p}_1(\Delta\tilde{x}) = g_{\tilde{\tau}}(\Delta\tilde{x}) \int_{-\frac{1}{2}}^{\frac{1}{2}} B(\Delta\tilde{x} + \tilde{x}_0)\, \mathrm{d}\tilde{x}_0 = g_{\tilde{\tau}}(\Delta\tilde{x})\, (1 - |\Delta\tilde{x}|)\, B(\Delta\tilde{x}/2);$$

this is a Gaussian multiplied by the "wedge" function $(1-|\Delta\tilde{x}|)$ and restricted to arguments in the range $-1 < \Delta\tilde{x} < 1$.

The two other parts can be simplified by moving the $\tilde{x}_0$-dependence in $B(\Delta\tilde{x} + \tilde{x}_0)$ into the integration limits combined with an external factor $B(\Delta\tilde{x}/2)$ as in $\bar{p}_1(\Delta\tilde{x})$, since only arguments with $-\frac{1}{2} < \Delta\tilde{x} + \tilde{x}_0 < \frac{1}{2}$, i.e. $-\frac{1}{2} - \Delta\tilde{x} < \tilde{x}_0 < \frac{1}{2} - \Delta\tilde{x}$ contribute to the integral:

$$
\begin{aligned}
\bar{p}_{2,3}(\Delta\tilde{x}) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} g_{\tilde{\tau}}(\Delta\tilde{x} \mp 1 + 2\tilde{x}_0)\; B(\Delta\tilde{x} + \tilde{x}_0)\; \mathrm{d}\tilde{x}_0 \\
&= B\Big(\frac{\Delta\tilde{x}}{2}\Big) \int_{\max(-\frac{1}{2}, -\frac{1}{2}-\Delta\tilde{x})}^{\min(\frac{1}{2}, \frac{1}{2}-\Delta\tilde{x})} g_{\tilde{\tau}}(\Delta\tilde{x} \mp 1 + 2\tilde{x}_0)\; \mathrm{d}\tilde{x}_0 \\
&= B\Big(\frac{\Delta\tilde{x}}{2}\Big) \int_{-\frac{1}{2}-\min(0,\Delta\tilde{x})}^{\frac{1}{2}-\max(0,\Delta\tilde{x})} g_{\tilde{\tau}}(\Delta\tilde{x} \mp 1 + 2\tilde{x}_0)\; \mathrm{d}\tilde{x}_0.
\end{aligned}
$$

If we call the integral $\int_{-\infty}^{x} g_{\tilde{\tau}}(x')\,\mathrm{d}x' = G_{\tilde{\tau}}(x)$, substitute $z = \Delta\tilde{x} \mp 1 + 2\tilde{x}_0$, and separate the two cases $\Delta\tilde{x} \geq 0$ and $\Delta\tilde{x} < 0$, this is

$$
\begin{aligned}
\bar{p}_2(\Delta\tilde{x}) &= B\Big(\frac{\Delta\tilde{x}}{2}\Big)\; \frac{G_{\tilde{\tau}}(z)}{2}\Big|_{\Delta\tilde{x}-1+2(-\frac{1}{2}-\min(0,\Delta\tilde{x}))}^{\Delta\tilde{x}-1+2(\frac{1}{2}-\max(0,\Delta\tilde{x}))} \\
&= \begin{cases} B\big(\frac{\Delta\tilde{x}}{2}\big)\; \frac{G_{\tilde{\tau}}(z)}{2}\Big|_{\Delta\tilde{x}-2}^{-\Delta\tilde{x}}, & \text{for } \Delta\tilde{x} \geq 0 \\ B\big(\frac{\Delta\tilde{x}}{2}\big)\; \frac{G_{\tilde{\tau}}(z)}{2}\Big|_{-\Delta\tilde{x}-2}^{\Delta\tilde{x}}, & \text{for } \Delta\tilde{x} < 0 \end{cases} \\
&= \begin{cases} \frac{1}{2}B\big(\frac{\Delta\tilde{x}}{2}\big)\big(G_{\tilde{\tau}}(-\Delta\tilde{x}) - G_{\tilde{\tau}}(\Delta\tilde{x} - 2)\big), & \text{for } \Delta\tilde{x} \geq 0 \\ \frac{1}{2}B\big(\frac{\Delta\tilde{x}}{2}\big)(G_{\tilde{\tau}}(\Delta\tilde{x}) - G_{\tilde{\tau}}(-\Delta\tilde{x} - 2)), & \text{for } \Delta\tilde{x} < 0 \end{cases} \\
&= \frac{1}{2}B\Big(\frac{\Delta\tilde{x}}{2}\Big)\Big(G_{\tilde{\tau}}(-|\Delta\tilde{x}|) - G_{\tilde{\tau}}(|\Delta\tilde{x}| - 2)\Big).
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{p}_3(\Delta\tilde{x}) &= B\Big(\frac{\Delta\tilde{x}}{2}\Big)\; \frac{G_{\tilde{\tau}}(z)}{2}\Big|_{\Delta\tilde{x}+1+2(-\frac{1}{2}-\min(0,\Delta\tilde{x}))}^{\Delta\tilde{x}+1+2(\frac{1}{2}-\max(0,\Delta\tilde{x}))} \\
&= \begin{cases} \frac{1}{2}B\big(\frac{\Delta\tilde{x}}{2}\big)\big(G_{\tilde{\tau}}(-\Delta\tilde{x} + 2) - G_{\tilde{\tau}}(\Delta\tilde{x})\big), & \text{for } \Delta\tilde{x} \geq 0 \\ \frac{1}{2}B\big(\frac{\Delta\tilde{x}}{2}\big)(G_{\tilde{\tau}}(\Delta\tilde{x} + 2) - G_{\tilde{\tau}}(-\Delta\tilde{x})), & \text{for } \Delta\tilde{x} < 0 \end{cases} \\
&= \frac{1}{2}B\Big(\frac{\Delta\tilde{x}}{2}\Big)\Big(G_{\tilde{\tau}}(-|\Delta\tilde{x}| + 2) - G_{\tilde{\tau}}(|\Delta\tilde{x}|)\Big).
\end{aligned}
$$

$G_{\tilde{\tau}}(z)$ is the integral of the (normalized, central) Gaussian probability distribution; hence $G_{\tilde{\tau}}(-z) = 1 - G_{\tilde{\tau}}(z)$ and $G_{\tilde{\tau}}(-z_1) - G_{\tilde{\tau}}(-z_2) = G_{\tilde{\tau}}(z_2) - G_{\tilde{\tau}}(z_1)$. Thus, the sum of $\bar{p}_2(\Delta\tilde{x})$ and $\bar{p}_3(\Delta\tilde{x})$ is

$$
\begin{aligned}
&\bar{p}_2(\Delta\tilde{x}) + \bar{p}_3(\Delta\tilde{x}) \\
&\quad = \frac{1}{2}B\Big(\frac{\Delta\tilde{x}}{2}\Big)\Big(G_{\tilde{\tau}}(-|\Delta\tilde{x}|) - G_{\tilde{\tau}}(|\Delta\tilde{x}| - 2) + G_{\tilde{\tau}}(-|\Delta\tilde{x}| + 2) - G_{\tilde{\tau}}(|\Delta\tilde{x}|)\Big) \\
&\quad = B\Big(\frac{\Delta\tilde{x}}{2}\Big)\Big(G_{\tilde{\tau}}(-|\Delta\tilde{x}|) - G_{\tilde{\tau}}(|\Delta\tilde{x}| - 2)\Big)
\end{aligned}
$$

and the complete probability distribution is

$$\bar{p}(\Delta\tilde{x}) = B\Big(\frac{\Delta\tilde{x}}{2}\Big)\Big(g_{\tilde{\tau}}(\Delta\tilde{x})\,(1-|\Delta\tilde{x}|) + \big(G_{\tilde{\tau}}(-|\Delta\tilde{x}|) - G_{\tilde{\tau}}(|\Delta\tilde{x}|-2)\big)\Big).$$

The effective diffusion coefficient corresponding to this distribution can be approximated (cf. Eq. (1)) as

$$\begin{aligned}
\tilde{D}_{\mathrm{eff}}(\tilde{\tau}) &\approx \frac{1}{2\tilde{\tau}}\int_{-\infty}^{\infty}(\Delta\tilde{x})^2\bar{p}(\Delta\tilde{x},\tilde{\tau})\,\mathrm{d}\Delta\tilde{x}\\
&= \frac{1}{2\tilde{\tau}}\int_{-1}^{1}(\Delta\tilde{x})^2\Big(g_{\tilde{\tau}}(\Delta\tilde{x})\,(1-|\Delta\tilde{x}|) + G_{\tilde{\tau}}(-|\Delta\tilde{x}|) - G_{\tilde{\tau}}(|\Delta\tilde{x}|-2)\Big)\,\mathrm{d}\Delta\tilde{x}\\
&= \frac{1}{\tilde{\tau}}\int_{0}^{1}(\Delta\tilde{x})^2\Big(g_{\tilde{\tau}}(\Delta\tilde{x})\,(1-\Delta\tilde{x}) + G_{\tilde{\tau}}(-\Delta\tilde{x}) - G_{\tilde{\tau}}(\Delta\tilde{x}-2)\Big)\,\mathrm{d}\Delta\tilde{x}\\
&= \frac{1}{\tilde{\tau}}\Big(\int_0^1(\Delta\tilde{x})^2 g_{\tilde{\tau}}(\Delta\tilde{x})\,\mathrm{d}\Delta\tilde{x} - \int_0^1(\Delta\tilde{x})^3 g_{\tilde{\tau}}(\Delta\tilde{x})\,\mathrm{d}\Delta\tilde{x}\\
&\qquad + \int_0^1(\Delta\tilde{x})^2 G_{\tilde{\tau}}(-\Delta\tilde{x})\,\mathrm{d}\Delta\tilde{x} - \int_0^1(\Delta\tilde{x})^2 G_{\tilde{\tau}}(\Delta\tilde{x}-2)\,\mathrm{d}\Delta\tilde{x}\Big)
\end{aligned}$$

Integrating by parts, we find $\int_0^1 z^2G(-z)\mathrm{d}z = \frac{z^3}{3}G(-z)|_0^1 + \int_0^1\frac{z^3}{3}g(z)\mathrm{d}z = \frac{G(-1)}{3} + \frac{1}{3}\int_0^1 z^3g(z)\mathrm{d}z$ and $\int_0^1 z^2G(z-2)\mathrm{d}z = \frac{z^3}{3}G(z-2)|_0^1 - \int_0^1\frac{z^3}{3}g(z-2)\mathrm{d}z = \frac{G(-1)}{3} - \frac{1}{3}\int_0^1 z^3g(z-2)\mathrm{d}z$, i. e.

$$\begin{aligned}
\tilde{D}_{\mathrm{eff}}(\tilde{\tau}) &\approx \frac{1}{\tilde{\tau}}\Big(\int_0^1(\Delta\tilde{x})^2 g_{\tilde{\tau}}(\Delta\tilde{x})\,\mathrm{d}\Delta\tilde{x} - \int_0^1(\Delta\tilde{x})^3 g_{\tilde{\tau}}(\Delta\tilde{x})\,\mathrm{d}\Delta\tilde{x}\\
&\qquad + \frac{1}{3}\int_0^1(\Delta\tilde{x})^3 g_{\tilde{\tau}}(\Delta\tilde{x})\,\mathrm{d}\Delta\tilde{x} + \frac{1}{3}\int_0^1(\Delta\tilde{x})^3 g_{\tilde{\tau}}(\Delta\tilde{x}-2)\,\mathrm{d}\Delta\tilde{x}\Big)\\
&= \frac{1}{\tilde{\tau}}\Big(\int_0^1(\Delta\tilde{x})^2 g_{\tilde{\tau}}(\Delta\tilde{x})\,\mathrm{d}\Delta\tilde{x} - \frac{2}{3}\int_0^1(\Delta\tilde{x})^3 g_{\tilde{\tau}}(\Delta\tilde{x})\,\mathrm{d}\Delta\tilde{x}\\
&\qquad + \frac{1}{3}\int_0^1(\Delta\tilde{x})^3 g_{\tilde{\tau}}(\Delta\tilde{x}-2)\,\mathrm{d}\Delta\tilde{x}\Big).
\end{aligned}$$

For sufficiently small $\tilde{\tau} \ll 1$ (i. e., the approximation for which our initial ansatz was valid), the third integral is very close to zero, since the normal distribution function is evaluated for arguments $|\Delta\tilde{x}| > 1 \gg \sqrt{\tilde{\tau}}$, and can, thus, be neglected. In addition, a further approximation at very small $\tilde{\tau}$ can be obtained by observing that the Gaussian distributions $g_{\tilde{\tau}}$ tend to zero quickly and, thus, the integrals can be calculated with an upper limit of ∞ instead of 1.[4] The first two integrals are therefore approximately

$$\int_0^{\infty}(\Delta\tilde{x})^2 g_{\tilde{\tau}}(\Delta\tilde{x})\,\mathrm{d}\Delta\tilde{x} = \Big[-\tilde{\tau}\Delta\tilde{x}g_{\tilde{\tau}}(\Delta\tilde{x}) + \tilde{\tau}(G_{\tilde{\tau}}(\Delta\tilde{x}) - \frac{1}{2})\Big]_0^{\infty} = \tilde{\tau} - \frac{1}{2}\tilde{\tau} = \frac{\tilde{\tau}}{2}$$

[4] Without this last approximation and using some basic integrals such as $\int\exp(-x^2)\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}\mathrm{erf}(x)$, $\int g_{\tilde{\tau}}(x)\,\mathrm{d}x = \int\frac{1}{\sqrt{2\pi\tilde{\tau}}}\exp(\frac{-x^2}{2\tilde{\tau}})\,\mathrm{d}x = \frac{1}{2}\mathrm{erf}(\frac{x}{\sqrt{2\tilde{\tau}}}) = G_{\tilde{\tau}}(x) - \frac{1}{2}$, $\int x\exp(-x^2)\,\mathrm{d}x = \frac{-1}{2}\exp(-x^2)$, $\int xg_{\tilde{\tau}}(x)\,\mathrm{d}x = \int\frac{x}{\sqrt{2\pi\tilde{\tau}}}\exp(\frac{-x^2}{2\tilde{\tau}})\,\mathrm{d}x = -\tilde{\tau}g_{\tilde{\tau}}(x)$, $\int x^2\exp(-x^2)\,\mathrm{d}x = \frac{-x}{2}\exp(-x^2) + \frac{\sqrt{\pi}}{4}\mathrm{erf}(x)$, $\int x^2g_{\tilde{\tau}}(x)\,\mathrm{d}x = \frac{-\tilde{\tau}x}{\sqrt{2\pi\tilde{\tau}}}\exp(-\frac{x^2}{2\tilde{\tau}}) + \frac{\tilde{\tau}}{2}\mathrm{erf}(\frac{x}{\sqrt{2\tilde{\tau}}}) = -\tilde{\tau}xg_{\tilde{\tau}}(x) + \tilde{\tau}(G_{\tilde{\tau}}(x) - \frac{1}{2})$, $\int x^3\exp(-x^2)\,\mathrm{d}x =$

and

$$\int_0^\infty (\Delta\tilde{x})^3 g_{\tilde{\tau}}(\Delta\tilde{x})\,\mathrm{d}\Delta\tilde{x} = \left[(-\tilde{\tau}(\Delta\tilde{x})^2 - 2\tilde{\tau}^2)g_{\tilde{\tau}}(\Delta\tilde{x})\right]_0^\infty = 2\tilde{\tau}^2 g_{\tilde{\tau}}(0) = \frac{2\tilde{\tau}^2}{\sqrt{2\pi\tilde{\tau}}} = \sqrt{\frac{2}{\pi}}\sqrt{\tilde{\tau}}^3,$$

which leads to the first result:

$$\tilde{D}_{\text{eff}}(\sqrt{\tilde{\tau}}) \approx \frac{1}{\tilde{\tau}}\Big(\frac{\tilde{\tau}}{2} - \frac{2}{3}\sqrt{\frac{2}{\pi}}\sqrt{\tilde{\tau}}^3\Big) = \frac{1}{2} - \frac{2}{3}\sqrt{\frac{2}{\pi}}\sqrt{\tilde{\tau}} \approx 0.5 - 0.531\,923\sqrt{\tilde{\tau}}. \qquad \text{(A.1)}$$

Appendix B. Calculation 2: Initial slope for $\tilde{P} > 0$

The assessment of the properties of the effective diffusion coefficient for short diffusion times in the presence of permeable membranes can be based on Eq. (3.16) in the article by Powles et al. [1992], which reads

$$p_{\text{P}}(x, x_0, \tau) = \begin{cases} g_\tau(x_0 + x) + g_\tau(x_0 - x) \\ \qquad -2P\int_0^\infty \mathrm{d}x' \exp(-2Px')g_\tau(x_0 - x + x'), & x < 0 \\ 2P\int_0^\infty \mathrm{d}x' \exp(-2Px')g_\tau(x_0 + x + x'), & x > 0, \end{cases}$$

and describes diffusion in the presence of a single barrier at $x_{\text{B}} = 0$ for an initial position of $-x_0$ [sic!], where $x_0 > 0$. First, we change x_0 into $-x_0$, which is a more convenient

$\frac{-x^2-1}{2}\exp(-x^2)$, $\int x^3 g_{\tilde{\tau}}(x)\,\mathrm{d}x = \frac{-\tilde{\tau}x^2-2\tilde{\tau}^2}{\sqrt{2\pi\tilde{\tau}}}\exp(-\frac{x^2}{2\tilde{\tau}}) = (-\tilde{\tau}x^2 - 2\tilde{\tau}^2)g_{\tilde{\tau}}(x)$, the first two integrals are

$$\int_0^1 (\Delta\tilde{x})^2 g_{\tilde{\tau}}(\Delta\tilde{x})\,\mathrm{d}\Delta\tilde{x} = \left[-\tilde{\tau}\Delta\tilde{x}g_{\tilde{\tau}}(\Delta\tilde{x}) + \tilde{\tau}(G_{\tilde{\tau}}(\Delta\tilde{x}) - \frac{1}{2})\right]_0^1 = -\tilde{\tau}g_{\tilde{\tau}}(1) + \tilde{\tau}G_{\tilde{\tau}}(1) - \tilde{\tau}G_{\tilde{\tau}}(0),$$

and

$$\int_0^1 (\Delta\tilde{x})^3 g_{\tilde{\tau}}(\Delta\tilde{x})\,\mathrm{d}\Delta\tilde{x} = \left[(-\tilde{\tau}(\Delta\tilde{x})^2 - 2\tilde{\tau}^2)g_{\tilde{\tau}}(\Delta\tilde{x})\right]_0^1 = (-\tilde{\tau} - 2\tilde{\tau}^2)g_{\tilde{\tau}}(1) + 2\tilde{\tau}^2 g_{\tilde{\tau}}(0).$$

Thus, collecting all terms, we find

$$\begin{aligned} D_{\text{eff}} &\approx \frac{1}{\tilde{\tau}}\Big(-\tilde{\tau}g_{\tilde{\tau}}(1) + \tilde{\tau}G_{\tilde{\tau}}(1) - \tilde{\tau}G_{\tilde{\tau}}(0) - \frac{2}{3}(-\tilde{\tau} - 2\tilde{\tau}^2)g_{\tilde{\tau}}(1) - \frac{4}{3}\tilde{\tau}^2 g_{\tilde{\tau}}(0)\Big) \\ &= -g_{\tilde{\tau}}(1) + G_{\tilde{\tau}}(1) - G_{\tilde{\tau}}(0) + \frac{2}{3}(1 + 2\tilde{\tau})g_{\tilde{\tau}}(1) - \frac{4}{3}\tilde{\tau}g_{\tilde{\tau}}(0) \\ &= -\frac{4}{3}\tilde{\tau}g_{\tilde{\tau}}(0) - \frac{1}{3}(1 - \tilde{\tau})g_{\tilde{\tau}}(1) - G_{\tilde{\tau}}(0) + G_{\tilde{\tau}}(1). \end{aligned}$$

In the limit $\tilde{\tau} \to 0$, all terms linear in $\tilde{\tau}$ vanish and $g_{\tilde{\tau}}(1) \to 0$, $G_{\tilde{\tau}}(1) \to 1$, and $G_{\tilde{\tau}}(0) = \frac{1}{2}$; hence, $D_{\text{eff}} \to \frac{1}{2}$ (as expected).

The derivative $\frac{\mathrm{d}D_{\text{eff}}}{\mathrm{d}\tilde{\tau}}$ is dominated by the first term, since the second one is quite close to zero due to $g_{\tilde{\tau}}(1)$ and the last term change only very little or not at all with $\tilde{\tau}$. The first term is

$$-\frac{4}{3}\tilde{\tau}g_{\tilde{\tau}}(0) = -\frac{4}{3}\frac{\tilde{\tau}}{\sqrt{2\pi\tilde{\tau}}} = -\frac{4}{3}\frac{1}{\sqrt{2\pi}}\sqrt{\tilde{\tau}}$$

and hence, the slope with respect to the (distribution's) standard deviation $\tilde{\sigma} = \sqrt{\tilde{\tau}}$ is

$$\frac{\mathrm{d}\tilde{D}_{\text{eff}}}{\mathrm{d}\tilde{\sigma}} = -\frac{4}{3}\frac{1}{\sqrt{2\pi}} = -\frac{2}{3}\sqrt{\frac{2}{\pi}} \approx -0.531\,923.$$

description of the starting point. Then, we use the scaled quantities of dimension 1 (which does not change anything in this equation) and move the barrier from $\tilde{x}_{\mathrm{B}} = 0$ to $\tilde{x}_{\mathrm{B}} = \frac{1}{2}$, i. e. replace $\tilde{x}$ and $\tilde{x}_0$ by $\tilde{x} - \frac{1}{2}$ and $\tilde{x}_0 - \frac{1}{2}$, respectively:

$$\tilde{p}(\tilde{x}, \tilde{x}_0, \tilde{\tau}, \tilde{P}) = \begin{cases} \tilde{p}_{\mathrm{L}}(\tilde{x}, \tilde{x}_0, \tilde{\tau}, \tilde{P}) = g_{\tilde{\tau}}(\tilde{x} - \tilde{x}_0) + g_{\tilde{\tau}}\big(1 - (\tilde{x} + \tilde{x}_0)\big) & \\ \quad - 2\tilde{P} \int_0^\infty \mathrm{d}\tilde{x}' \exp(-2\tilde{P}\tilde{x}') g_{\tilde{\tau}}(1 - (\tilde{x} + \tilde{x}_0) + \tilde{x}'), & \tilde{x} < \frac{1}{2} \\ \tilde{p}_{\mathrm{R}}(\tilde{x}, \tilde{x}_0, \tilde{\tau}, \tilde{P}) = & \\ \quad 2\tilde{P} \int_0^\infty \mathrm{d}\tilde{x}' \exp(-2\tilde{P}\tilde{x}') g_{\tilde{\tau}}(\tilde{x} - \tilde{x}_0 + \tilde{x}'), & \tilde{x} \geq \frac{1}{2}. \end{cases} \tag{B.1}$$

Now, we combine two such probability distributions to model barriers at $\tilde{x}_{\mathrm{B}} = +\frac{1}{2}$ and $\tilde{x}_{\mathrm{B}} = -\frac{1}{2}$; this is an approximation only valid for very narrow distributions, i. e., for $\tilde{\tau} \ll 1$, where at most one barrier is "seen" by any diffusing particle:

$$\tilde{p}_{\mathrm{sym}}(\tilde{x}, \tilde{x}_0, \tilde{\tau}, \tilde{P}) = \begin{cases} \tilde{p}(\tilde{x}, \tilde{x}_0, \tilde{\tau}, \tilde{P}), & x_0 \geq 0 \\ \tilde{p}(-\tilde{x}, -\tilde{x}_0, \tilde{\tau}, \tilde{P}), & x_0 < 0 \end{cases}$$

As before, this distribution has to be averaged over all initial positions $-\frac{1}{2} < \tilde{x}_0 < \frac{1}{2}$:

$$\begin{aligned} \bar{p}(\Delta\tilde{x}, \tilde{\tau}, \tilde{P}) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{p}_{\mathrm{sym}}(\tilde{x}_0 + \Delta\tilde{x}, \tilde{x}_0, \tilde{\tau}, \tilde{P})\, \mathrm{d}\tilde{x}_0 \\ &= \int_{-\frac{1}{2}}^{0} \tilde{p}_{\mathrm{sym}}(\tilde{x}_0 + \Delta\tilde{x}, \tilde{x}_0, \tilde{\tau}, \tilde{P})\, \mathrm{d}\tilde{x}_0 + \int_{0}^{\frac{1}{2}} \tilde{p}_{\mathrm{sym}}(\tilde{x}_0 + \Delta\tilde{x}, \tilde{x}_0, \tilde{\tau}, \tilde{P})\, \mathrm{d}\tilde{x}_0 \\ &= \underbrace{\int_{-\frac{1}{2}}^{0} \tilde{p}(-\tilde{x}_0 - \Delta\tilde{x}, -\tilde{x}_0, \tilde{\tau}, \tilde{P})\, \mathrm{d}\tilde{x}_0}_{=\bar{p}_-(\Delta\tilde{x}, \tilde{\tau}, \tilde{P})} + \underbrace{\int_{0}^{\frac{1}{2}} \tilde{p}(\tilde{x}_0 + \Delta\tilde{x}, \tilde{x}_0, \tilde{\tau}, \tilde{P})\, \mathrm{d}\tilde{x}_0}_{=\bar{p}_+(\Delta\tilde{x}, \tilde{\tau}, \tilde{P})}. \end{aligned}$$

It is sufficient to calculate the first of these integrals ($\tilde{x}_0 > 0$), i. e., $\bar{p}_+(\Delta\tilde{x}, \tilde{\tau}, \tilde{P})$, since $\bar{p}_-(\Delta\tilde{x}, \tilde{\tau}, \tilde{P}) = \bar{p}_+(-\Delta\tilde{x}, \tilde{\tau}, \tilde{P})$; the final function $\bar{p}$ can be obtained by symmetrizing with respect to $\Delta\tilde{x}$:

$$\bar{p}(\Delta\tilde{x}, \tilde{\tau}, \tilde{P}) = \bar{p}_+(-\Delta\tilde{x}, \tilde{\tau}, \tilde{P}) + \bar{p}_+(\Delta\tilde{x}, \tilde{\tau}, \tilde{P}).$$

The integral $\bar{p}_+(\Delta\tilde{x}, \tilde{\tau}, \tilde{P})$ can be calculated by inserting Eq. (B.1) depending on the value of $\tilde{x} = \tilde{x}_0 + \Delta\tilde{x}$ (differentiate between $\tilde{x}_0 + \Delta\tilde{x} < \frac{1}{2}$ and $\tilde{x}_0 + \Delta\tilde{x} \geq \frac{1}{2}$, i. e., based on the value of $\Delta\tilde{x}$ since $\tilde{x}_0$ is the integration variable). This can be done by separating the two cases $\Delta\tilde{x} < 0$ (which implies $\tilde{x} < 0$, i. e., the first case of Eq. (B.1) for all values of $\tilde{x}_0$) and $\Delta\tilde{x} \geq 0$ (here, both cases of Eq. (B.1) have to be considered depending on the value of the integration variable $\tilde{x}_0$):

$$\bar{p}_+(\Delta\tilde{x},\tilde{\tau},\tilde{P}) = \int_0^{\frac{1}{2}} \tilde{p}(\tilde{x}_0+\Delta\tilde{x},\tilde{x}_0,\tilde{\tau},\tilde{P})\,\mathrm{d}\tilde{x}_0$$
$$= \begin{cases} \int_0^{\frac{1}{2}-\Delta\tilde{x}} \tilde{p}_\mathrm{L}(\tilde{x}_0+\Delta\tilde{x},\tilde{x}_0,\tilde{\tau},\tilde{P})\,\mathrm{d}\tilde{x}_0 \\ \qquad + \int_{\frac{1}{2}-\Delta\tilde{x}}^{\frac{1}{2}} \tilde{p}_\mathrm{R}(\tilde{x}_0+\Delta\tilde{x},\tilde{x}_0,\tilde{\tau},\tilde{P})\,\mathrm{d}\tilde{x}_0, & 0 \le \Delta\tilde{x} < \frac{1}{2} \\ \int_0^{\frac{1}{2}} \tilde{p}_\mathrm{L}(\tilde{x}_0+\Delta\tilde{x},\tilde{x}_0,\tilde{\tau},\tilde{P})\,\mathrm{d}\tilde{x}_0, & -\frac{1}{2} < \Delta\tilde{x} < 0 \end{cases}$$

The case $|\Delta\tilde{x}| \ge \frac{1}{2}$ can be neglected, since only very narrow distributions $\tilde{p}$ are considered, i. e., $\bar{p}_+(\Delta\tilde{x},\tilde{\tau},\tilde{P}) \approx 0$ for $|\Delta\tilde{x}| > \frac{1}{2}$.

B.1. Negative $\Delta\tilde{x}$

For $\Delta\tilde{x} < 0$, we obtain

$$\begin{aligned}
\bar{p}_+(\Delta\tilde{x},\tilde{\tau},\tilde{P}) &= \int_0^{\frac{1}{2}} \tilde{p}_\mathrm{L}(\tilde{x}_0+\Delta\tilde{x},\tilde{x}_0,\tilde{\tau},\tilde{P})\,\mathrm{d}\tilde{x}_0 \\
&= \int_0^{\frac{1}{2}} \mathrm{d}\tilde{x}_0 \Big[g_{\tilde{\tau}}(\Delta\tilde{x}) + g_{\tilde{\tau}}(1-2\tilde{x}_0-\Delta\tilde{x}) \\
&\qquad - 2\tilde{P}\int_0^\infty \mathrm{d}\tilde{x}'\exp(-2\tilde{P}\tilde{x}')g_{\tilde{\tau}}(1-2\tilde{x}_0-\Delta\tilde{x}+\tilde{x}')\Big] \\
&= \frac{1}{2}g_{\tilde{\tau}}(\Delta\tilde{x}) + \int_0^{\frac{1}{2}} \mathrm{d}\tilde{x}_0\, g_{\tilde{\tau}}(1-2\tilde{x}_0-\Delta\tilde{x}) \\
&\qquad - 2\tilde{P}\int_0^\infty \mathrm{d}\tilde{x}' \left[\exp(-2\tilde{P}\tilde{x}')\int_0^{\frac{1}{2}} \mathrm{d}\tilde{x}_0\, g_{\tilde{\tau}}(1-2\tilde{x}_0-\Delta\tilde{x}+\tilde{x}')\right] \\
&= \frac{1}{2}g_{\tilde{\tau}}(\Delta\tilde{x}) - \left[\frac{1}{2}G_{\tilde{\tau}}(1-2\tilde{x}_0-\Delta\tilde{x})\right]_0^{\frac{1}{2}} \\
&\qquad + 2\tilde{P}\int_0^\infty \mathrm{d}\tilde{x}' \left(\exp(-2\tilde{P}\tilde{x}')\left[\frac{1}{2}G_{\tilde{\tau}}(1-2\tilde{x}_0-\Delta\tilde{x}+\tilde{x}')\right]_0^{\frac{1}{2}}\right) \\
&= \frac{1}{2}g_{\tilde{\tau}}(\Delta\tilde{x}) - \frac{1}{2}G_{\tilde{\tau}}(-\Delta\tilde{x}) + \frac{1}{2}G_{\tilde{\tau}}(1-\Delta\tilde{x}) \\
&\qquad + \tilde{P}\int_0^\infty \mathrm{d}\tilde{x}'\exp(-2\tilde{P}\tilde{x}')\Big(G_{\tilde{\tau}}(-\Delta\tilde{x}+\tilde{x}') - G_{\tilde{\tau}}(1-\Delta\tilde{x}+\tilde{x}')\Big)
\end{aligned}$$

For a narrow distribution, $\Delta\tilde{x} \geq -\frac{1}{2}$, and $\tilde{x}' > 0$, we can approximate $G_{\tilde{\tau}}(1-\Delta\tilde{x}) \approx 1$ and $G_{\tilde{\tau}}(1-\Delta\tilde{x}+x') \approx 1$, hence

$$\begin{aligned}
\bar{p}_+(\Delta\tilde{x},\tilde{\tau},\tilde{P}) \approx \frac{1}{2} g_{\tilde{\tau}}(\Delta\tilde{x}) - \frac{1}{2} G_{\tilde{\tau}}(-\Delta\tilde{x}) + \frac{1}{2} \\
+ \tilde{P}\int_0^\infty \mathrm{d}\tilde{x}' \exp(-2\tilde{P}\tilde{x}') G_{\tilde{\tau}}(-\Delta\tilde{x}+\tilde{x}') - \underbrace{\tilde{P}\int_0^\infty \mathrm{d}\tilde{x}' \exp(-2\tilde{P}\tilde{x}')}_{=\frac{1}{2}} \\
= \frac{1}{2} g_{\tilde{\tau}}(\Delta\tilde{x}) - \frac{1}{2} G_{\tilde{\tau}}(-\Delta\tilde{x}) + \tilde{P}\int_0^\infty \mathrm{d}\tilde{x}' \exp(-2\tilde{P}\tilde{x}') G_{\tilde{\tau}}(-\Delta\tilde{x}+\tilde{x}').
\end{aligned}$$

The last summand can be calculated (similarly as above) using integration by parts and the integral of $\exp(-(ax+bx^2))$

$$\begin{aligned}
&\int_0^\infty \big(\tilde{P}\exp(-2\tilde{P}\tilde{x}')\big)\, G_{\tilde{\tau}}(-\Delta\tilde{x}+\tilde{x}')\,\mathrm{d}\tilde{x}' \\
&= \left[\frac{-1}{2}\exp(-2\tilde{P}\tilde{x}') G_{\tilde{\tau}}(-\Delta\tilde{x}+\tilde{x}')\right]_0^\infty - \int_0^\infty \big(\frac{-1}{2}\exp(-2\tilde{P}\tilde{x}')\big)\, g_{\tilde{\tau}}(-\Delta\tilde{x}+\tilde{x}')\,\mathrm{d}\tilde{x}' \\
&= \frac{1}{2} G_{\tilde{\tau}}(-\Delta\tilde{x}) + \frac{1}{2\sqrt{2\pi\tilde{\tau}}}\int_0^\infty \exp\Big(-2\tilde{P}\tilde{x}' - \frac{(-\Delta\tilde{x}+\tilde{x}')^2}{2\tilde{\tau}}\Big)\,\mathrm{d}\tilde{x}' \\
&= \frac{1}{2} G_{\tilde{\tau}}(-\Delta\tilde{x}) + \frac{1}{2\sqrt{2\pi\tilde{\tau}}}\exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big)\int_0^\infty \exp\Big(-\Big((2\tilde{P}-\frac{\Delta\tilde{x}}{\tilde{\tau}})\tilde{x}' + \frac{1}{2\tilde{\tau}}\tilde{x}'^2\Big)\Big)\,\mathrm{d}\tilde{x}' \\
&= \frac{1}{2} G_{\tilde{\tau}}(-\Delta\tilde{x}) + \frac{1}{2\sqrt{2\pi\tilde{\tau}}}\exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big)\frac{\sqrt{2\pi\tilde{\tau}}}{2}\exp\Big(\frac{\tilde{\tau}}{2}(2\tilde{P}-\frac{\Delta\tilde{x}}{\tilde{\tau}})^2\Big) \\
&\qquad \cdot\left[\mathrm{erf}\Big(\frac{\tilde{x}'}{\sqrt{2\tilde{\tau}}} + \sqrt{\frac{\tilde{\tau}}{2}}(2\tilde{P}-\frac{\Delta\tilde{x}}{\tilde{\tau}})\Big)\right]_0^\infty \\
&= \frac{1}{2} G_{\tilde{\tau}}(-\Delta\tilde{x}) + \frac{1}{4}\exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big)\exp\Big(\Big(\frac{2\tilde{\tau}\tilde{P}-\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big)^2\Big)\mathrm{erfc}\Big(\frac{2\tilde{\tau}\tilde{P}-\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big) \\
&= \frac{1}{2} G_{\tilde{\tau}}(-\Delta\tilde{x}) + \frac{1}{4}\exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big)\mathcal{E}\Big(\frac{2\tilde{\tau}\tilde{P}-\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big),
\end{aligned}$$

where $\mathcal{E}(x) = \exp(x^2)\mathrm{erfc}(x)$. The final result for $\Delta\tilde{x} < 0$ is therefore

$$\bar{p}_+(\Delta\tilde{x},\tilde{\tau},\tilde{P}) = \frac{1}{2} g_{\tilde{\tau}}(\Delta\tilde{x}) + \frac{1}{4}\exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big)\mathcal{E}\Big(\frac{2\tilde{\tau}\tilde{P}-\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big). \tag{B.2}$$

B.2. Positive $\Delta\tilde{x}$

For $\Delta\tilde{x} \geq 0$, we obtain

$$\bar{p}_+(\Delta\tilde{x}, \tilde{\tau}, \tilde{P}) = \int_0^{\frac{1}{2}-\Delta\tilde{x}} \tilde{p}_{\mathrm{L}}(\tilde{x}_0 + \Delta\tilde{x}, \tilde{x}_0, \tilde{\tau}, \tilde{P})\,\mathrm{d}\tilde{x}_0 + \int_{\frac{1}{2}-\Delta\tilde{x}}^{\frac{1}{2}} \tilde{p}_{\mathrm{R}}(\tilde{x}_0 + \Delta\tilde{x}, \tilde{x}_0, \tilde{\tau}, \tilde{P})\,\mathrm{d}\tilde{x}_0$$

$$= \int_0^{\frac{1}{2}-\Delta\tilde{x}} \mathrm{d}\tilde{x}_0 \left[g_{\tilde{\tau}}(\Delta\tilde{x}) + g_{\tilde{\tau}}(1-2\tilde{x}_0-\Delta\tilde{x}) - 2\tilde{P}\int_0^\infty \mathrm{d}\tilde{x}' \exp(-2\tilde{P}\tilde{x}') g_{\tilde{\tau}}(1-2\tilde{x}_0-\Delta\tilde{x}+\tilde{x}') \right]$$
$$+ \int_{\frac{1}{2}-\Delta\tilde{x}}^{\frac{1}{2}} \mathrm{d}\tilde{x}_0 \left[2\tilde{P}\int_0^\infty \mathrm{d}\tilde{x}' \exp(-2\tilde{P}\tilde{x}') g_{\tilde{\tau}}(\Delta\tilde{x}+\tilde{x}') \right]. \quad \text{(B.3)}$$

The first integral is similar to the one for $\Delta\tilde{x} < 0$ but with a different upper integration limit. Following a similar calculation as for $\Delta\tilde{x} < 0$, this is

$$\int_0^{\frac{1}{2}-\Delta\tilde{x}} \mathrm{d}\tilde{x}_0 \left[g_{\tilde{\tau}}(\Delta\tilde{x}) + g_{\tilde{\tau}}(1-2\tilde{x}_0-\Delta\tilde{x}) - 2\tilde{P}\int_0^\infty \mathrm{d}\tilde{x}' \exp(-2\tilde{P}\tilde{x}') g_{\tilde{\tau}}(1-2\tilde{x}_0-\Delta\tilde{x}+\tilde{x}') \right]$$
$$= (\frac{1}{2} - \Delta\tilde{x}) g_{\tilde{\tau}}(\Delta\tilde{x}) - \frac{1}{2}\Big[G_{\tilde{\tau}}(1-2\tilde{x}_0-\Delta\tilde{x}) \Big]_0^{\frac{1}{2}-\Delta\tilde{x}}$$
$$- 2\tilde{P}\int_0^\infty \mathrm{d}\tilde{x}' \left[\exp(-2\tilde{P}\tilde{x}') \int_0^{\frac{1}{2}-\Delta\tilde{x}} \mathrm{d}\tilde{x}_0\, g_{\tilde{\tau}}(1-2\tilde{x}_0-\Delta\tilde{x}+\tilde{x}') \right]$$
$$= (\frac{1}{2} - \Delta\tilde{x}) g_{\tilde{\tau}}(\Delta\tilde{x}) - \frac{1}{2}\Big(G_{\tilde{\tau}}(\Delta\tilde{x}) - G_{\tilde{\tau}}(1-\Delta\tilde{x}) \Big)$$
$$+ \tilde{P}\int_0^\infty \exp(-2\tilde{P}\tilde{x}') \Big[G_{\tilde{\tau}}(1-2\tilde{x}_0-\Delta\tilde{x}+\tilde{x}') \Big]_0^{\frac{1}{2}-\Delta\tilde{x}} \mathrm{d}\tilde{x}'$$
$$\approx (\frac{1}{2} - \Delta\tilde{x}) g_{\tilde{\tau}}(\Delta\tilde{x}) - \frac{1}{2} G_{\tilde{\tau}}(\Delta\tilde{x}) + \frac{1}{2} + \tilde{P}\int_0^\infty \exp(-2\tilde{P}\tilde{x}') \big(G_{\tilde{\tau}}(\Delta\tilde{x}+\tilde{x}') - 1 \big) \mathrm{d}\tilde{x}'$$
$$= (\frac{1}{2} - \Delta\tilde{x}) g_{\tilde{\tau}}(\Delta\tilde{x}) - \frac{1}{2} G_{\tilde{\tau}}(\Delta\tilde{x}) + \int_0^\infty (\tilde{P}\exp(-2\tilde{P}\tilde{x}')) G_{\tilde{\tau}}(\Delta\tilde{x}+\tilde{x}')\,\mathrm{d}\tilde{x}'.$$

The final integral can again be integrated by parts

$$\int_0^\infty (\tilde{P}\exp(-2\tilde{P}\tilde{x}'))G_{\tilde{\tau}}(\Delta\tilde{x}+\tilde{x}')\,\mathrm{d}\tilde{x}'$$

$$= \Big[\frac{-1}{2}\exp(-2\tilde{P}\tilde{x}')G_{\tilde{\tau}}(\Delta\tilde{x}+\tilde{x}')\Big]_0^\infty - \int_0^\infty \big(\frac{-1}{2}\exp(-2\tilde{P}\tilde{x}')\big)g_{\tilde{\tau}}(\Delta\tilde{x}+\tilde{x}')\,\mathrm{d}\tilde{x}'$$

$$= \frac{1}{2}G_{\tilde{\tau}}(\Delta\tilde{x}) + \frac{1}{2\sqrt{2\pi\tilde{\tau}}}\int_0^\infty \exp\Big(-2\tilde{P}\tilde{x}' - \frac{(\Delta\tilde{x}+\tilde{x}')^2}{2\tilde{\tau}}\Big)\mathrm{d}\tilde{x}'$$

$$= \frac{1}{2}G_{\tilde{\tau}}(\Delta\tilde{x}) + \frac{1}{2\sqrt{2\pi\tilde{\tau}}}\exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big)\int_0^\infty \exp\Big(-\Big((2\tilde{P}+\frac{\Delta\tilde{x}}{\tilde{\tau}})\tilde{x}' + \frac{1}{2\tilde{\tau}}\tilde{x}'^2\Big)\Big)\mathrm{d}\tilde{x}'$$

$$= \frac{1}{2}G_{\tilde{\tau}}(\Delta\tilde{x}) + \frac{1}{2\sqrt{2\pi\tilde{\tau}}}\exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big)\frac{\sqrt{2\pi\tilde{\tau}}}{2}\exp\Big(\Big(\frac{2\tilde{\tau}\tilde{P}+\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big)^2\Big) \cdot \Big[\mathrm{erf}\Big(\frac{\tilde{x}'}{\sqrt{2\tilde{\tau}}} + \frac{2\tilde{\tau}\tilde{P}+\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big)\Big]_0^\infty$$

$$= \frac{1}{2}G_{\tilde{\tau}}(\Delta\tilde{x}) + \frac{1}{4}\exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big)\exp\Big(\Big(\frac{2\tilde{\tau}\tilde{P}+\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big)^2\Big)\mathrm{erfc}\Big(\frac{2\tilde{\tau}\tilde{P}+\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big)$$

$$= \frac{1}{2}G_{\tilde{\tau}}(\Delta\tilde{x}) + \frac{1}{4}\exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big)\mathcal{E}\Big(\frac{2\tilde{\tau}\tilde{P}+\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big).$$

The second integral in Eq. (B.3) is

$$\int_{\frac{1}{2}-\Delta\tilde{x}}^{\frac{1}{2}} \mathrm{d}\tilde{x}_0 \Big[2\tilde{P}\int_0^\infty \mathrm{d}\tilde{x}'\exp(-2\tilde{P}\tilde{x}')g_{\tilde{\tau}}(\Delta\tilde{x}+\tilde{x}')\Big]$$

$$= 2\tilde{P}\Delta\tilde{x}\int_0^\infty \exp(-2\tilde{P}\tilde{x}')g_{\tilde{\tau}}(\Delta\tilde{x}+\tilde{x}')\,\mathrm{d}\tilde{x}' = \frac{2\tilde{P}\Delta\tilde{x}}{\sqrt{2\pi\tilde{\tau}}}\int_0^\infty \exp\Big(-2\tilde{P}\tilde{x}' - \frac{(\Delta\tilde{x}+\tilde{x}')^2}{2\tilde{\tau}}\Big)\mathrm{d}\tilde{x}'$$

$$= \frac{2\tilde{P}\Delta\tilde{x}}{\sqrt{2\pi\tilde{\tau}}}\exp\big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\big)\int_0^\infty \exp\Big(-\Big((2\tilde{P}+\frac{\Delta\tilde{x}}{\tilde{\tau}})\tilde{x}' + \frac{1}{2\tilde{\tau}}\tilde{x}'^2\Big)\Big)\,\mathrm{d}\tilde{x}'$$

$$= \frac{2\tilde{P}\Delta\tilde{x}}{\sqrt{2\pi\tilde{\tau}}}\exp\big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\big)\frac{\sqrt{2\pi\tilde{\tau}}}{2}\exp\Big(\frac{\tilde{\tau}}{2}\big(2\tilde{P}+\frac{\Delta\tilde{x}}{\tilde{\tau}}\big)^2\Big)\Big[\mathrm{erf}\Big(\frac{\tilde{x}'}{\sqrt{2\tilde{\tau}}} + \sqrt{\frac{\tilde{\tau}}{2}}\big(2\tilde{P}+\frac{\Delta\tilde{x}}{\tilde{\tau}}\big)\Big)\Big]_0^\infty$$

$$= \tilde{P}\Delta\tilde{x}\exp\big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\big)\exp\Big(\Big(\frac{2\tilde{\tau}\tilde{P}+\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big)^2\Big)\mathrm{erfc}\Big(\frac{2\tilde{\tau}\tilde{P}+\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big)$$

$$= \tilde{P}\Delta\tilde{x}\exp\big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\big)\mathcal{E}\Big(\frac{2\tilde{\tau}\tilde{P}+\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big).$$

Collecting all terms, the final result of Eq. (B.3) for $\Delta\tilde{x} > 0$ is

$$\bar{p}_+(\Delta\tilde{x},\tilde{\tau},\tilde{P}) \approx$$

$$(\frac{1}{2}-\Delta\tilde{x})g_{\tilde{\tau}}(\Delta\tilde{x}) + \frac{1}{4}\exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big)\mathcal{E}\Big(\frac{2\tilde{\tau}\tilde{P}+\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big) + \tilde{P}\Delta\tilde{x}\exp\big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\big)\mathcal{E}\Big(\frac{2\tilde{\tau}\tilde{P}+\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big)$$

$$= (\frac{1}{2}-\Delta\tilde{x})g_{\tilde{\tau}}(\Delta\tilde{x}) + \Big(\frac{1}{4}+\tilde{P}\Delta\tilde{x}\Big)\exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big)\mathcal{E}\Big(\frac{2\tilde{\tau}\tilde{P}+\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big). \quad \text{(B.4)}$$

B.3. Symmetrized probability distribution

Summarizing the results of the preceding subsections (Eqs. (B.2) and (B.4)), we have

$$\bar{p}_+(\Delta\tilde{x},\tilde{\tau},\tilde{P}) \approx \begin{cases} (\frac{1}{2}-\Delta\tilde{x})g_{\tilde{\tau}}(\Delta\tilde{x}) + \left(\frac{1}{4}+\tilde{P}\Delta\tilde{x}\right)\exp\left(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\right)\mathcal{E}\left(\frac{2\tilde{\tau}\tilde{P}+\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\right), & \Delta\tilde{x}\geq 0 \\ \frac{1}{2}g_{\tilde{\tau}}(\Delta\tilde{x}) + \frac{1}{4}\exp\left(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\right)\mathcal{E}\left(\frac{2\tilde{\tau}\tilde{P}-\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\right), & \Delta\tilde{x}<0 \end{cases}$$

and, consequently,

$$\bar{p}_+(-\Delta\tilde{x},\tilde{\tau},\tilde{P}) \approx \begin{cases} \frac{1}{2}g_{\tilde{\tau}}(\Delta\tilde{x}) + \frac{1}{4}\exp\left(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\right)\mathcal{E}\left(\frac{2\tilde{\tau}\tilde{P}+\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\right), & \Delta\tilde{x}\geq 0 \\ (\frac{1}{2}+\Delta\tilde{x})g_{\tilde{\tau}}(\Delta\tilde{x}) + \left(\frac{1}{4}-\tilde{P}\Delta\tilde{x}\right)\exp\left(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\right)\mathcal{E}\left(\frac{2\tilde{\tau}\tilde{P}-\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\right), & \Delta\tilde{x}<0, \end{cases}$$

resulting in the symmetrized result

$$\begin{aligned} \bar{p}(\Delta\tilde{x},\tilde{\tau},\tilde{P}) &\approx \begin{cases} (1-\Delta\tilde{x})g_{\tilde{\tau}}(\Delta\tilde{x}) + \left(\frac{1}{2}+\tilde{P}\Delta\tilde{x}\right)\exp\left(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\right)\mathcal{E}\left(\frac{2\tilde{\tau}\tilde{P}+\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\right), & \Delta\tilde{x}\geq 0 \\ (1+\Delta\tilde{x})g_{\tilde{\tau}}(\Delta\tilde{x}) + \left(\frac{1}{2}-\tilde{P}\Delta\tilde{x}\right)\exp\left(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\right)\mathcal{E}\left(\frac{2\tilde{\tau}\tilde{P}-\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\right), & \Delta\tilde{x}<0 \end{cases} \\ &= (1-|\Delta\tilde{x}|)g_{\tilde{\tau}}(\Delta\tilde{x}) + \left(\frac{1}{2}+\tilde{P}|\Delta\tilde{x}|\right)\exp\left(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\right)\mathcal{E}\left(\frac{2\tilde{\tau}\tilde{P}+|\Delta\tilde{x}|}{\sqrt{2\tilde{\tau}}}\right). \end{aligned}$$

B.4. Variance of probability distribution

As before, we calculate now the effective diffusion coefficient using the variance of the probability distribution:

$$
\begin{aligned}
\tilde{D}_{\text{eff}}(\tilde{\tau}) &= \frac{1}{2\tilde{\tau}} \int_{-\infty}^{\infty} (\Delta\tilde{x})^2 \overline{p}(\Delta\tilde{x}, \tilde{\tau}, \tilde{P}) \,\mathrm{d}\Delta\tilde{x} \\
&\approx \frac{1}{2\tilde{\tau}} \int_{-\infty}^{\infty} (\Delta\tilde{x})^2 \Big[(1 - |\Delta\tilde{x}|) g_{\tilde{\tau}}(\Delta\tilde{x}) \\
&\qquad + \Big(\frac{1}{2} + \tilde{P}|\Delta\tilde{x}|\Big) \exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big) \mathcal{E}\Big(\frac{2\tilde{\tau}\tilde{P} + |\Delta\tilde{x}|}{\sqrt{2\tilde{\tau}}}\Big) \Big] \,\mathrm{d}\Delta\tilde{x} \\
&= \frac{1}{\tilde{\tau}} \int_{0}^{\infty} (\Delta\tilde{x})^2 \Big[(1 - \Delta\tilde{x}) g_{\tilde{\tau}}(\Delta\tilde{x}) \\
&\qquad + \Big(\frac{1}{2} + \tilde{P}\Delta\tilde{x}\Big) \exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big) \mathcal{E}\Big(\frac{2\tilde{\tau}\tilde{P} + \Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big) \Big] \,\mathrm{d}\Delta\tilde{x} \\
&= \frac{1}{\tilde{\tau}} \Bigg(\int_{0}^{\infty} (\Delta\tilde{x})^2 g_{\tilde{\tau}}(\Delta\tilde{x}) \,\mathrm{d}\Delta\tilde{x} - \int_{0}^{\infty} (\Delta\tilde{x})^3 g_{\tilde{\tau}}(\Delta\tilde{x}) \,\mathrm{d}\Delta\tilde{x} \\
&\qquad + \frac{1}{2} \int_{0}^{\infty} (\Delta\tilde{x})^2 \exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big) \mathcal{E}\Big(\frac{2\tilde{\tau}\tilde{P} + \Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big) \,\mathrm{d}\Delta\tilde{x} \\
&\qquad + \tilde{P} \int_{0}^{\infty} (\Delta\tilde{x})^3 \exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big) \mathcal{E}\Big(\frac{2\tilde{\tau}\tilde{P} + \Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big) \,\mathrm{d}\Delta\tilde{x} \Bigg) \\
&= \frac{1}{2} - \sqrt{\frac{2\tilde{\tau}}{\pi}} + \frac{1}{2\tilde{\tau}} \int_{0}^{\infty} (\Delta\tilde{x})^2 \exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big) \mathcal{E}\Big(\frac{2\tilde{\tau}\tilde{P} + \Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big) \,\mathrm{d}\Delta\tilde{x} \\
&\qquad + \frac{\tilde{P}}{\tilde{\tau}} \int_{0}^{\infty} (\Delta\tilde{x})^3 \exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big) \mathcal{E}\Big(\frac{2\tilde{\tau}\tilde{P} + \Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big) \,\mathrm{d}\Delta\tilde{x}.
\end{aligned}
$$

The first of the remaining integrals is

$$
\begin{aligned}
&\int_{0}^{\infty} (\Delta\tilde{x})^2 \exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big) \mathcal{E}\Big(\frac{2\tilde{\tau}\tilde{P} + \Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big) \,\mathrm{d}\Delta\tilde{x} \\
&\quad = \int_{0}^{\infty} (\Delta\tilde{x})^2 \exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big) \exp\Big(\frac{(2\tilde{\tau}\tilde{P} + \Delta\tilde{x})^2}{2\tilde{\tau}}\Big) \operatorname{erfc}\Big(\frac{2\tilde{\tau}\tilde{P} + \Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big) \,\mathrm{d}\Delta\tilde{x} \\
&\quad = \int_{0}^{\infty} (\Delta\tilde{x})^2 \exp\Big(\frac{4\tilde{\tau}^2\tilde{P}^2 + 4\tilde{\tau}\tilde{P}\Delta\tilde{x}}{2\tilde{\tau}}\Big) \operatorname{erfc}\Big(\frac{2\tilde{\tau}\tilde{P} + \Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big) \,\mathrm{d}\Delta\tilde{x} \\
&\quad = \exp(2\tilde{\tau}\tilde{P}^2) \int_{0}^{\infty} (\Delta\tilde{x})^2 \exp(2\tilde{P}\Delta\tilde{x}) \operatorname{erfc}\Big(\frac{2\tilde{\tau}\tilde{P} + \Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big) \,\mathrm{d}\Delta\tilde{x}.
\end{aligned}
$$

According to WolframAlpha[5], we have

$$\int x^2 \exp(2abx)\,\mathrm{erfc}(a+bx)\,\mathrm{d}x$$
$$= \frac{\exp(-a^2-b^2x^2)}{4a^3b^3}\Big((a^2+1)\exp(b^2x^2)\mathrm{erf}(bx)$$
$$+\exp\big((a+bx)^2\big)\,(2a^2b^2x^2-2abx+1)\mathrm{erfc}(a+bx) - \frac{2a}{\sqrt{\pi}}(abx-1)\Big)$$
$$= \frac{\exp(-a^2)}{4a^3b^3}\Big((a^2+1)\mathrm{erf}(bx)$$
$$+\exp(a^2+2abx)\,(2a^2b^2x^2-2abx+1)\mathrm{erfc}(a+bx) - \frac{2a}{\sqrt{\pi}}(abx-1)\exp(-b^2x^2)\Big)$$

and thus

$$\int_0^\infty x^2\exp(2abx)\,\mathrm{erfc}(a+bx)\,\mathrm{d}x = \frac{\exp(-a^2)}{4a^3b^3}\Big((a^2+1)-\exp(a^2)\mathrm{erfc}(a)-\frac{2a}{\sqrt{\pi}}\Big).$$

For $a=\tilde{P}\sqrt{2\tilde{\tau}}$ and $b=1/\sqrt{2\tilde{\tau}}$, this is

$$\int_0^\infty x^2\exp(2\tilde{P}x)\,\mathrm{erfc}\Big(\frac{2\tilde{\tau}\tilde{P}+x}{\sqrt{2\tilde{\tau}}}\Big)\,\mathrm{d}x$$
$$= \frac{\exp(-2\tilde{\tau}\tilde{P}^2)}{4\tilde{P}^3}\Big((2\tilde{\tau}\tilde{P}^2+1)-\exp(2\tilde{\tau}\tilde{P}^2)\mathrm{erfc}(\sqrt{2\tilde{\tau}}\tilde{P})-2\tilde{P}\sqrt{\frac{2\tilde{\tau}}{\pi}}\Big).$$

The second remaining integral is similar (but contains $\Delta\tilde{x}^3$ instead of $\Delta\tilde{x}^2$:

$$\int_0^\infty(\Delta\tilde{x})^3\exp\Big(\frac{-\Delta\tilde{x}^2}{2\tilde{\tau}}\Big)\mathcal{E}\Big(\frac{2\tilde{\tau}\tilde{P}+\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big)\,\mathrm{d}\Delta\tilde{x}$$
$$=\exp(2\tilde{\tau}\tilde{P}^2)\int_0^\infty(\Delta\tilde{x})^3\exp(2\tilde{P}\Delta\tilde{x})\mathrm{erfc}\Big(\frac{2\tilde{\tau}\tilde{P}+\Delta\tilde{x}}{\sqrt{2\tilde{\tau}}}\Big)\,\mathrm{d}\Delta\tilde{x}.$$

According to WolframAlpha[6], we have

$$\int x^3\exp(2abx)\,\mathrm{erfc}(a+bx)\,\mathrm{d}x$$
$$=\frac{\exp(-a^2-b^2x^2)}{8a^4b^4}\Big(-3(a^2+1)\exp(b^2x^2)\mathrm{erf}(bx)-\frac{2a}{\sqrt{\pi}}\Big(2a^2(b^2x^2+1)-3abx+3\Big)$$
$$+\exp\big((a+bx)^2\big)\,\big(4a^3b^3x^3-6a^2b^2x^2+6abx-3\big)\mathrm{erfc}(a+bx)\Big)$$
$$=\frac{\exp(-a^2)}{8a^4b^4}\Big(-3(a^2+1)\mathrm{erf}(bx)-\frac{2a}{\sqrt{\pi}}\Big(2a^2(b^2x^2+1)-3abx+3\Big)\exp(-b^2x^2)$$
$$+\exp(a^2+2abx)\,\big(4a^3b^3x^3-6a^2b^2x^2+6abx-3\big)\mathrm{erfc}(a+bx)\Big)$$

[5] `http://www.wolframalpha.com/input/?i=integral(x^2*exp(2*a*b*x)*erfc(a%2Bb*x))`

[6] `http://www.wolframalpha.com/input/?i=integral(x^3*exp(2*a*b*x)*erfc(a%2Bb*x))`

and thus

$$\int_0^\infty x^3 \exp(2abx)\,\mathrm{erfc}(a+bx)\,\mathrm{d}x$$
$$= \frac{\exp(-a^2)}{8a^4b^4}\Big(-3(a^2+1) + \frac{2a}{\sqrt{\pi}}(2a^2+3) + 3\exp(a^2)\mathrm{erfc}(a)\Big).$$

For $a = \tilde{P}\sqrt{2\tilde{\tau}}$ and $b = 1/\sqrt{2\tilde{\tau}}$, this is

$$\int_0^\infty x^3 \exp(2\tilde{P}x)\,\mathrm{erfc}\Big(\frac{2\tilde{\tau}\tilde{P}+x}{\sqrt{2\tilde{\tau}}}\Big)\,\mathrm{d}x$$
$$= \frac{\exp(-2\tilde{\tau}\tilde{P}^2)}{8\tilde{P}^4}\Big(-3(2\tilde{\tau}\tilde{P}^2+1) + 2\tilde{P}\sqrt{\frac{2\tilde{\tau}}{\pi}}(4\tilde{\tau}\tilde{P}^2+3) + 3\exp(2\tilde{\tau}\tilde{P}^2)\mathrm{erfc}(\sqrt{2\tilde{\tau}}\tilde{P})\Big).$$

The final result for $\tilde{D}_{\text{eff}}$ is therefore

$$\begin{aligned}
\tilde{D}_{\text{eff}}(\tilde{\tau}) &\approx \frac{1}{2} - \sqrt{\frac{2\tilde{\tau}}{\pi}} + \frac{1}{8\tilde{\tau}\tilde{P}^3}\Big((2\tilde{\tau}\tilde{P}^2+1) - \exp(2\tilde{\tau}\tilde{P}^2)\mathrm{erfc}(\sqrt{2\tilde{\tau}}\tilde{P}) - 2\tilde{P}\sqrt{\frac{2\tilde{\tau}}{\pi}}\Big) \\
&\quad + \frac{1}{8\tilde{\tau}\tilde{P}^3}\Big(-3(2\tilde{\tau}\tilde{P}^2+1) + 2\tilde{P}\sqrt{\frac{2\tilde{\tau}}{\pi}}(4\tilde{\tau}\tilde{P}^2+3) \\
&\qquad + 3\exp(2\tilde{\tau}\tilde{P}^2)\mathrm{erfc}(\sqrt{2\tilde{\tau}}\tilde{P})\Big) \\
&= \frac{1}{2} - \sqrt{\frac{2\tilde{\tau}}{\pi}} + \frac{1}{8\tilde{\tau}\tilde{P}^3}\Big(-4\tilde{\tau}\tilde{P}^2 - 2 \\
&\qquad + 4\tilde{P}(1+2\tilde{\tau}\tilde{P}^2)\sqrt{\frac{2\tilde{\tau}}{\pi}} + 2\exp(2\tilde{\tau}\tilde{P}^2)\mathrm{erfc}(\sqrt{2\tilde{\tau}}\tilde{P})\Big) \\
&= \frac{1}{2} - \sqrt{\frac{2\tilde{\tau}}{\pi}} - \frac{1}{2\tilde{P}} - \frac{1}{4\tilde{\tau}\tilde{P}^3} + \frac{1+2\tilde{\tau}\tilde{P}^2}{2\tilde{\tau}\tilde{P}^2}\sqrt{\frac{2\tilde{\tau}}{\pi}} \\
&\quad + \frac{1}{4\tilde{\tau}\tilde{P}^3}\exp(2\tilde{\tau}\tilde{P}^2)\mathrm{erfc}(\sqrt{2\tilde{\tau}}\tilde{P}) \\
&= \frac{1}{2} - \frac{1}{2\tilde{P}} - \frac{1}{4\tilde{\tau}\tilde{P}^3} + \frac{1}{\tilde{P}^2\sqrt{2\pi\tilde{\tau}}} + \frac{1}{4\tilde{\tau}\tilde{P}^3}\exp(2\tilde{\tau}\tilde{P}^2)\mathrm{erfc}(\sqrt{2\tilde{\tau}}\tilde{P}).
\end{aligned} \tag{B.5}$$

Based on the Taylor expansions of $\exp(ax) = 1 + ax + \frac{1}{2}(ax)^2 + \frac{1}{6}(ax)^3 + \frac{1}{24}(ax)^4 + \cdots$ and $\mathrm{erfc}(ax) = 1 - \frac{2}{\sqrt{\pi}}ax + \frac{2}{3\sqrt{\pi}}(ax)^3 - \frac{1}{5\sqrt{\pi}}(ax)^5 + \frac{1}{21\sqrt{\pi}}(ax)^7 - + \cdots$ [7], this can be

[7] Alternatively, the following result can be calculated directly from the series expansion of the product $\mathrm{erfc}(\sqrt{2x}a)\exp(2xa^2)/(4xa^3)$ at $x = 0$, which is (according to WolframAlpha)

$$\frac{1}{4a^3x} - \frac{1}{a^2\sqrt{2\pi x}} + \frac{1}{2a} - \frac{2}{3}\sqrt{\frac{2}{\pi}}\sqrt{x} + \frac{a}{2}x - \frac{8}{15}\sqrt{\frac{2}{\pi}}a^2x^{3/2} + \frac{a^3}{3}x^2 - \frac{32}{105}\sqrt{\frac{2}{\pi}}a^4x^{5/2} + \frac{a^5}{6}x^3 - + \cdots$$

(`http://www.wolframalpha.com/input/?i=series+expansion+erfc(sqrt(2x)a)*exp(2xa^2)/(4xa^3)+at+x=0`).

approximated for very small $\tilde{\tau}$ as

$$
\begin{aligned}
\tilde{D}_{\text{eff}}(\sqrt{\tilde{\tau}}) \approx & \frac{1}{2} - \frac{1}{2\tilde{P}} - \frac{1}{4\tilde{\tau}\tilde{P}^3} + \frac{1}{\tilde{P}^2\sqrt{2\pi\tilde{\tau}}} \\
& + \frac{1}{4\tilde{\tau}\tilde{P}^3}\Big(1 + 2\tilde{\tau}\tilde{P}^2 + 2\tilde{\tau}^2\tilde{P}^4 + \cdots\Big)\Big(1 - \frac{2}{\sqrt{\pi}}\sqrt{2\tilde{\tau}}\,\tilde{P} + \frac{2}{3\sqrt{\pi}}(\sqrt{2\tilde{\tau}}\tilde{P})^3 - + \cdots\Big) \\
& \approx \frac{1}{2} - \frac{2}{3}\sqrt{\frac{2}{\pi}}\sqrt{\tilde{\tau}} + \frac{\tilde{P}}{2}\tilde{\tau} - \frac{8}{15}\sqrt{\frac{2}{\pi}}\tilde{P}^2\,\tilde{\tau}^{3/2} + \frac{\tilde{P}^3}{3}\tilde{\tau}^2 - \frac{32}{105}\sqrt{\frac{2}{\pi}}\tilde{P}^4\,\tilde{\tau}^{5/2} + \frac{\tilde{P}^5}{6}\tilde{\tau}^3. \quad \text{(B.6)}
\end{aligned}
$$

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Manuscript version history

2014-11-12 first version

2014-11-28 update:

- added better approximation based on the product $\exp(2\tilde{\tau}\tilde{P}^2)\mathrm{erfc}(\sqrt{2\tilde{\tau}}\tilde{P})$
- added power-series approximation to 6th order in $\sqrt{\tilde{\tau}}$
- added 2 figures to illustrate methods and results

2014-12-01 correction in caption of Fig. 1: $\tilde{x}_0$ instead of x_0