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or
Choosing the optimal T_R in averaged acquisitions

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1 The problem: How many "averages"?

Consider the following simple question in magnetic resonance imaging (MRI) or spectroscopy (MRS): Given a fixed total measurement time, T_total, (e. g., a typical breath-hold duration of 16 s or a maximum accepted sequence duration of 10 min) and the possibility to fit into this duration an acquisition with several repeated read-outs (that are to be averaged to increase the data quality), what is the optimum balance between the repetition time (TR or T_R) (limiting the longitudinal relaxation) and the number of averaged signals (also called simply the number of averages, N, or the number of excitations, NEX or N_ex)?

In the following analysis, we consider only pulse sequences with 90° excitations (e. g., spin-echo sequences), in which all available magnetization is flipped to the transverse plain in each repetition. The analysis of fast spoiled gradient-echo sequences with smaller

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flip angles (FLASH technique) is more complex due to the relations between the T_R, the optimal flip angle (Ernst angle), the optimal read-out duration (or receiver bandwidth), and the transverse relaxation.

2 The solution (for 90° excitations)

In general, a single acquisition (without averaging) requires M repeated excitations separated by T_R (e. g., M = "number of phase-encoding steps" for conventional spin-echo acquisitions or $M = 1$ for single-shot EPI or one-dimensional MR spectroscopy). Hence, the acquisition time without data averaging is $T_{acq} = MT_R$, and, neglecting for now that the number of averages must be integer, this number of averages is $N = T_{total}/T_{acq} = T_{total}/(MT_R)$. In the following, the actual relevant time parameter is, thus, $T_{avail} = T_{total}/M$, the time available for each required excitation, and the number of averages is $N = T_{avail}/T_R$.

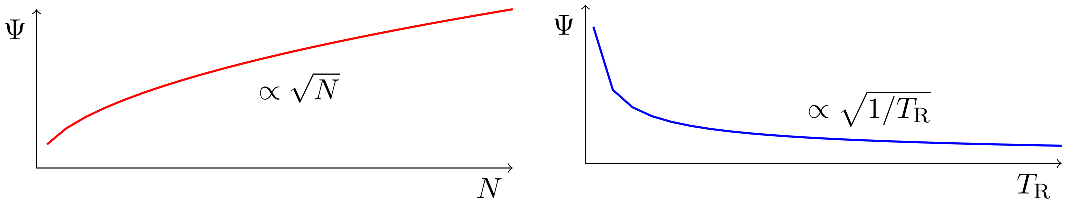
We can also express the repetition time by these parameters as

$$T_R = T_{acq}/M = T_{total}/(MN) = T_{avail}/N. \tag{1}$$

2.1 Exact solution

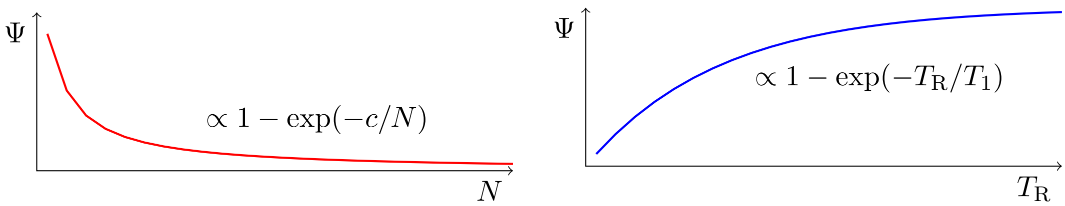
On the one hand, averaging data increases the obtainable signal-to-noise ratio (SNR or Ψ) proportional to the square root of the number of averages:

$$\Psi \propto \sqrt{N} = \sqrt{T_{avail}/T_R}. \tag{2}$$



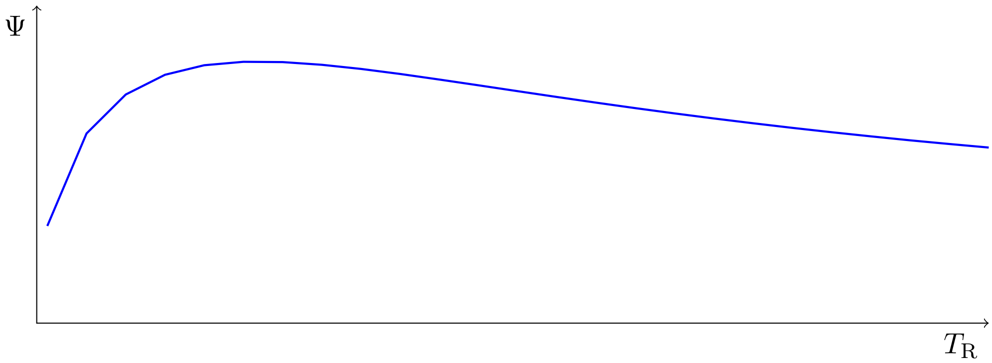
On the other hand, increasing the number of averages (at fixed total measurement duration!) results in a shortened time for longitudinal relaxation: $T_R = T_{avail}/N$ and, thus, in less SNR:

$$\Psi \propto 1 - \exp(-T_R/T_1) = 1 - \exp\left(\frac{-T_{avail}}{NT_1}\right). \tag{3}$$



The resulting total SNR is proportional to the product of both factors, i. e.,

$$\Psi \propto \sqrt{T_{\text{avail}}/T_{\text{R}}}\Big(1 - \exp(-T_{\text{R}}/T_1)\Big) : \tag{4}$$



The SNR (as a function of T_{R} at constant T_{avail}) has, thus, a maximum in an intermediate range; for very short T_{R} (and, consequently, many averages), the longitudinal magnetization cannot relax sufficiently, which reduces the available signal; for very long T_{R}, the relaxation is approximately complete anyway, but the number of averages decreases.

To calculate the optimum T_{R} (or N), T_{R} and T_{avail} are best expressed in terms of T_1 using $\tau = T_{\text{R}}/T_1$ and $T = T_{\text{avail}}/T_1$, which gives

$$\Psi \propto \sqrt{\frac{T}{\tau}}\Big(1 - \exp(-\tau)\Big) \tag{5}$$

The maximum of this expression with respect to τ is obtained by setting its derivative to zero, i. e.

$$0 = \frac{\partial}{\partial\tau}\left[\sqrt{\frac{T}{\tau}}\Big(1 - \exp(-\tau)\Big)\right] = \frac{\exp(-\tau)\sqrt{T}}{2\sqrt{\tau^3}}\Big(1 + 2\tau - \exp(\tau)\Big) \tag{6}$$

which gives

$$1 + 2\tau - \exp(\tau) = 0. \tag{7}$$

The solution of this equation is

$$\tau = -\frac{1}{2} - W_{-1}\left(\frac{-1}{2\sqrt{e}}\right) \approx 1.256\,431\,208\,626\ldots \tag{8}$$

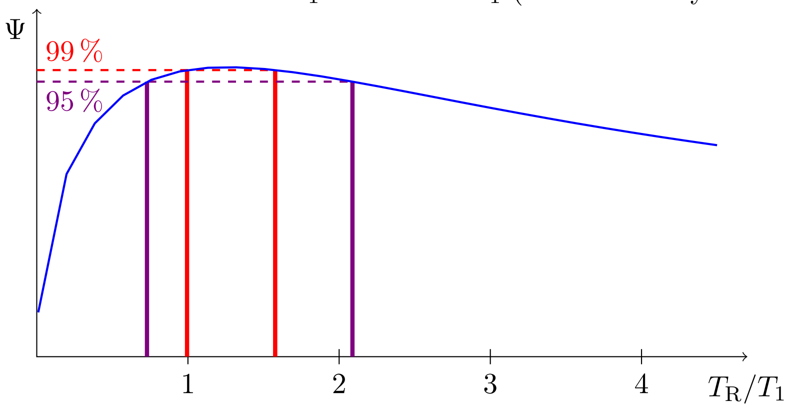
where W_{-1} is the lower branch of the Lambert W-function[1].

[1] The Lambert W-function (or product-log or Omega function), $W(z)$, gives the principal solution for W in $z = W\exp(W)$, i. e., it is the inverse function of $f(W) = W\exp(W)$. Since f is not injective, $W(z)$ is multi-valued and, in particular, when restricting to real-valued W, double-valued on the interval $(-1/e, 0)$; on this interval, the upper branch ($W \geq -1$) is denoted $W_0(z)$ and the lower branch ($W \leq -1$) is denoted $W_{-1}(z)$.

Hence, the optimal choice for T_R is (at least theoretically),

$$T_\mathrm{R} \approx 1.256\,T_1. \tag{9}$$

Fortunately, the function to be maximized has a rather broad maximum, and one obtains values between 99 % and 100 % of the maximum SNR for T_R between about $0.9937\,T_1$ and $1.5773\,T_1$ and values between 95 % and 100 % of the maximum SNR for T_R between about $0.7293\,T_1$ and $2.0882\,T_1$ (determined by numerical evaluation).

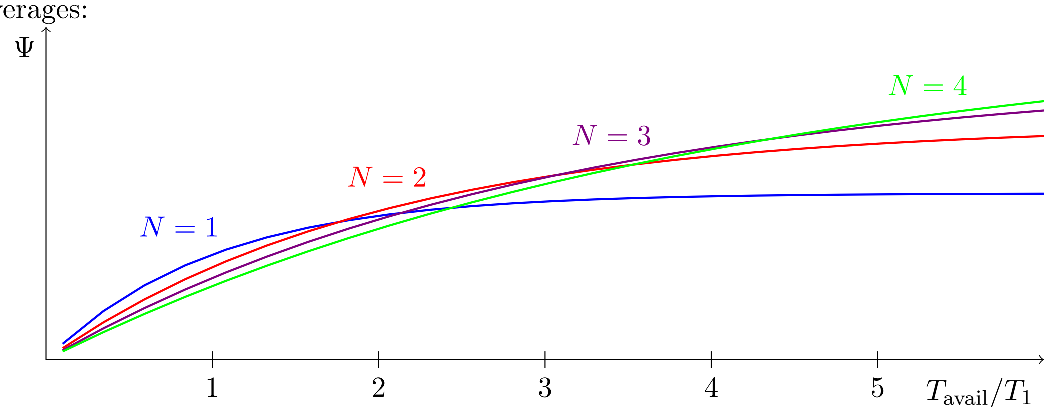


So, practically, choosing T_R to be T_1 gives still a nearly optimal SNR; if a larger number of signal acquisitions is preferred, e. g., for statistical evaluation, then T_R may even be shortened to $0.75\,T_1$, and if – on the other hand – the base SNR of every single acquisition becomes very low (but is to be reconstructed before signal averaging) then T_R may be chosen to be $2\,T_1$ to increase the quality of each single (not averaged) data set.

2.2 Special cases (small discrete N)

The evaluation above is based on the assumption that many averages can be obtained (or that N is considered as a (pseudo)continuous variable). Thus, in the case of very few averages (e. g., $N < 10$), a slightly different approach might be appropriate.

The problem is illustrated below by the SNR dependence on T_avail/T_1 for $N = 1, 2, 3, 4$ averages:



Obviously, $N = 1$ (with the longest possible T_R) is the optimal choice for small values of T_avail/T_1, but instead of increasing T_R to very long values it is better to reduce T_R and increase the number of averages at a certain point.

This critical point is always reached, if T_R is shorter than about $1.256\,T_1$ for N averages, but longer for $N-1$ averages.

2.2.1 $N = 1$ or $N = 2$?

If the available measurement time T_avail is such that the choice is between only 1 or 2 averages (with a T_R of T_avail or $T_\mathrm{avail}/2$, respectively), then the SNR is equal for both strategies if

$$1 - \exp(-\tau) = \sqrt{2}\,\bigl(1 - \exp(-\tau/2)\bigr). \tag{10}$$

This equation is equivalent to a quadratic equation that can be obtained by substituting $x = \exp(-\tau/2)$:

$$x^2 - \sqrt{2}\,x + (\sqrt{2} - 1) = 0 \tag{11}$$

with the solution

$$x = \frac{1 \pm \sqrt{3 - 2\sqrt{2}}}{\sqrt{2}} \tag{12}$$

i. e.,

$$\tau = 2\ln\left(\frac{\sqrt{2}}{1 - \sqrt{3 - 2\sqrt{2}}}\right) \approx 1.7627 \tag{13}$$

Thus, higher SNR is obtained with $N = 1$ if T_avail is shorter than about $1.763\,T_1$ (so, e. g., for $T_\mathrm{avail} = 1.6\,T_1$, use $N = 1, T_\mathrm{R} = 1.6\ T_1$ but not $N = 2, T_\mathrm{R} = 0.8\,T_1$). If T_avail becomes longer, then use $N = 2$ (e. g., for $T_\mathrm{avail} = 1.8\,T_1$, $N = 2, T_\mathrm{R} = 0.9\,T_1$ instead of $N = 1, T_\mathrm{R} = 1.8\,T_1$). This is approximately consistent with choosing T_R such that it is closest to the magic number $1.256\,T_1$.

2.2.2 $N = 2$ or $N = 3$?

In this case, equal SNR is obtained if

$$\sqrt{2}\bigl(1 - \exp(-\tau/2)\bigr) = \sqrt{3}\,\bigl(1 - \exp(-\tau/3)\bigr). \tag{14}$$

This equation is equivalent to a cubic equation for $x = \exp(-\tau/6)$. A numerical solution yields

$$\tau \approx 3.069\,14 \tag{15}$$

i. e., use 2 averages if $T_\mathrm{avail} < 3.07\,T_1$ (the resulting T_R is then lower than $1.535\,T_1$) and 3 averages if $T_\mathrm{avail} > 3.07\,T_1$ (i. e., $T_\mathrm{R} > 1.023\,T_1$).

2.2.3 $N = 3$ or $N = 4$?

In this case, equal SNR is obtained if

$$\sqrt{3}\big(1 - \exp(-\tau/3)\big) = \sqrt{4}\,\big(1 - \exp(-\tau/4)\big). \tag{16}$$

A numerical solution yields

$$\tau \approx 4.346\,35 \tag{17}$$

i. e., use 3 averages if the $T_{\text{avail}} < 4.35\,T_1$ ($T_\text{R} < 1.449\,T_1$) and 4 averages if $T_{\text{avail}} > 4.35\,T_1$ ($T_\text{R} > 1.087\,T_1$).

3 Concluding comments

The presented results are not new; in fact, it has been derived before (at least numerically) in several publications. An early publication is, e. g., R. R. Ernst and R. E. Morgan, "Saturation effects in Fourier spectroscopy", *Molecular Physics* 1973; 26(1): 49–74.

The presented analysis is valid only for simple 90°excitations; FLASH sequences or steady-state pulse sequences which refocus the transverse magnetization as well will exhibit a different behavior.

Although the number of averages is at first sight a clearly discrete parameter, it can be varied almost continuously in many situations, e. g., by changing the oversampling parameters. If oversampling is set to 50 % (instead of 0) in a spin-echo or gradient-echo sequence, this is equivalent to a virtual N of 1.5. (This is not correct in single-shot sequences such as EPI where relaxation effects during the echo train have to be considered.) Thus, an optimal balance between T_R and N with $T_\text{R} \approx 1.256\,T_1$ is possible in many cases.

Of course, optimizing T_R as described above is *not* an option if T_1-weighted or T_2-weighted images are to be acquired. In these cases, the optimal value of T_R depends on the desired contrast and not on the total SNR.