What's special about 1.256 431 208 626... in MRI?

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Consider the following simple question in magnetic resonance imaging (MRI) or spectroscopy (MRS): Given a fixed total measurement time, T_{total}, (e. g., a typical breath-hold duration of 16 s or a maximum accepted sequence duration of 10 min) and the possibility to fit into this duration an acquisition with several repeated read-outs (that are to be averaged to increase the data quality), what is the optimum balance between the repetition time (TR or T_{R}) and the number of averaged signals (also called simply the number of averages, N, or the number of excitations, NEX or N_{ex})? (For simplicity, let's consider only pulse sequences with 90 ° excitations (e. g., spin-echo sequences), in which all available magnetization is flipped to the transverse plain in each repetition.)

1 The solution (for 90° excitations)

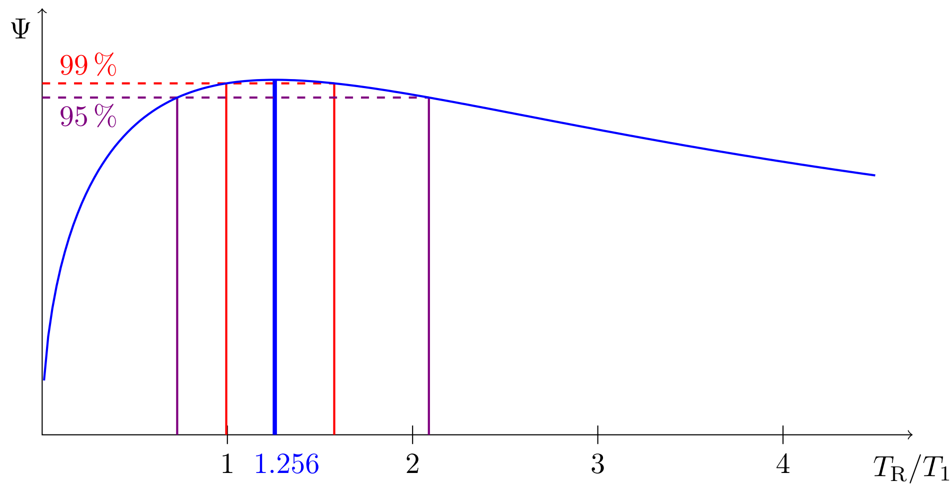

Figure 1: Dependence of the signal-to-noise ratio Ψ on the repetition time T_{R} at fixed total measurement time with a maximum at $T_{\text{R}}/T_1 = 1.256\,431\,208\,626\ldots$

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In general, a single MRI acquisition (without averaging) requires M repeated excitations separated by T_R (e. g., M could be the number of phase-encoding steps for conventional spin-echo acquisitions or $M = 1$ for single-shot EPI or one-dimensional MR spectroscopy). Hence, the acquisition time without data averaging is $T_\mathrm{acq} = MT_\mathrm{R}$, and, neglecting for now that the number of averages must be integer, this number of averages is $N = T_\mathrm{total}/T_\mathrm{acq} = T_\mathrm{total}/(MT_\mathrm{R})$. In the following, the actual relevant time parameter is, thus, $T_\mathrm{avail} = T_\mathrm{total}/M$, the time available for each required excitation, and the number of averages is $N = T_\mathrm{avail}/T_\mathrm{R}$. We can also express the repetition time by these parameters as:

$$T_\mathrm{R} = T_\mathrm{acq}/M = T_\mathrm{total}/(MN) = T_\mathrm{avail}/N. \tag{1}$$

On the one hand, averaging data increases the obtainable signal-to-noise ratio (SNR or Ψ) proportional to the square root of the number of averages:

$$\Psi \propto \sqrt{N} = \sqrt{T_\mathrm{avail}/T_\mathrm{R}}. \tag{2}$$

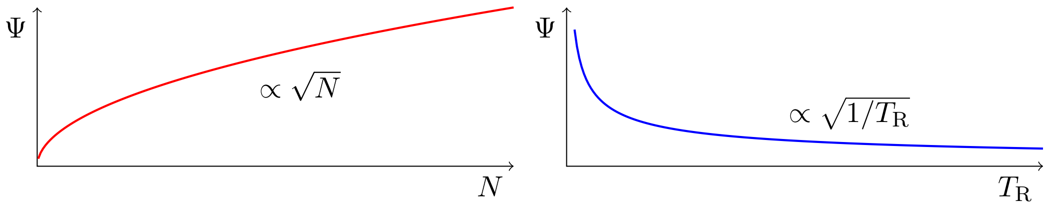

Figure 2: Dependence of the SNR Ψ on the number N of averages (left) and the T_R (right, at fixed total measurement time) due to averaging neglecting the influence of T_R on T_1 relaxation

On the other hand, increasing the number of averages (at fixed total measurement duration!) results in a shortened time for longitudinal relaxation: $T_\mathrm{R} = T_\mathrm{avail}/N$ and, thus, in less SNR:

$$\Psi \propto 1 - \exp(-T_\mathrm{R}/T_1) = 1 - \exp \frac{-T_\mathrm{avail}}{NT_1}. \tag{3}$$

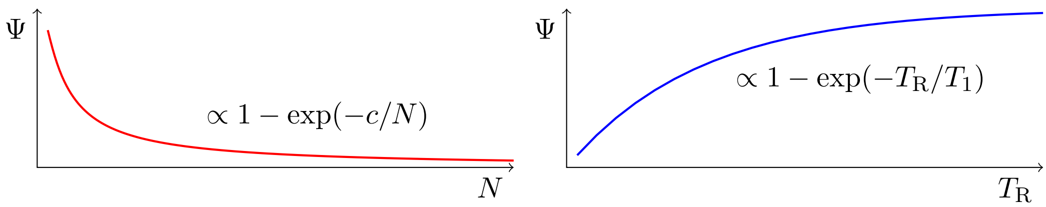

Figure 3: Dependence of the SNR Ψ on the number N of averages (left, at fixed total measurement time) and the T_R (right) due to T_1 relaxation neglecting the influence of signal averaging

The resulting total SNR is proportional to the product of both factors, i. e.,

$$\Psi \propto \sqrt{T_{\text{avail}}/T_{\text{R}}}(1 - \exp(-T_{\text{R}}/T_1)) : \tag{4}$$

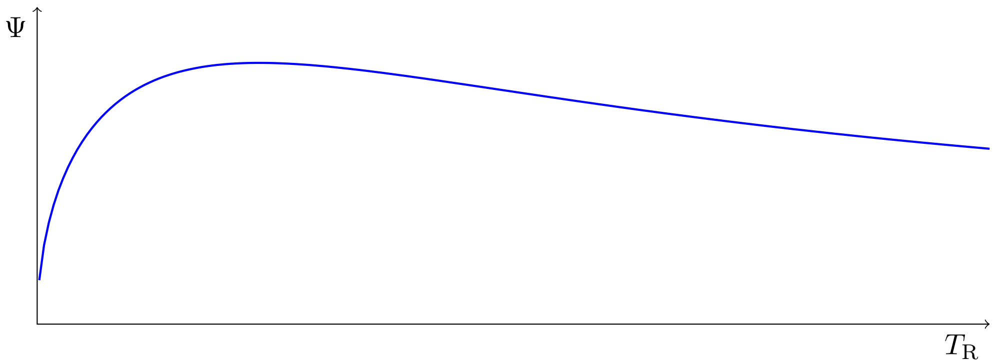

Figure 4: Resulting dependence of the SNR Ψ on T_{R} at constant total measurement time (and thus, implicitly, on the number of averages) considering both averaging and relaxation effects

The SNR (as a function of T_{R} at constant total measurement time) has, thus, a maximum in an intermediate range; for very short T_{R} (and, consequently, many averages), the longitudinal magnetization cannot relax sufficiently, which reduces the available signal; for very long T_{R}, the relaxation is approximately complete anyway, but the number of averages decreases.

To calculate the optimum T_{R} (or N), T_{R} and T_{avail} are best expressed in terms of T_1 using $\tau = T_{\text{R}}/T_1$ and $T = T_{\text{avail}}/T_1$, which gives

$$\Psi \propto \sqrt{\frac{T}{\tau}(1 - \exp(-\tau))} \tag{5}$$

The maximum of this expression with respect to τ is obtained by setting its derivative to zero, i. e.

$$0 = \frac{\partial}{\partial \tau}[\sqrt{T/\tau}(1 - \exp(-\tau))] = \frac{\exp(-\tau)}{2}\sqrt{\frac{T}{\tau^3}}(1 + 2\tau - \exp(\tau)) \tag{6}$$

which gives

$$1 + 2\tau - \exp(\tau) = 0. \tag{7}$$

The solution of this equation is

$$\tau = -\frac{1}{2} - W_{-1}(-1/(2\sqrt{e})) = 1.256\,431\,208\,626\ldots \tag{8}$$

where W_{-1} is the lower branch of the Lambert W-function (or product-log or Omega function, which gives the solution for W in $z = W\exp(W)$, i. e., it is the inverse function of $f(W) = W\exp(W)$). Hence, the optimal choice for T_R is (at least theoretically):

$$T_\mathrm{R} \approx 1.256\,T_1. \tag{9}$$

Fortunately, the function to be maximized has a rather broad maximum, and one obtains values between 99 % and 100 % of the maximum SNR for T_R between about $0.9937\,T_1$ and $1.5773\,T_1$ and values between 95 % and 100 % of the maximum SNR for T_R between about $0.7293\,T_1$ and $2.0882\,T_1$ (values determined by numerical evaluation and illustrated in the first figure at the top).

So, practically, choosing T_R to be T_1 gives still a nearly optimal SNR; if a larger number of signal acquisitions is preferred, e. g., for statistical evaluation, then T_R may even be shortened to $0.75\,T_1$, and if – on the other hand – the base SNR of every single acquisition becomes very low (but is to be reconstructed before signal averaging) then T_R may be chosen to be $2\,T_1$ to increase the quality of each single (not averaged) data set.

The presented result is not new; in fact, it has been derived before (at least numerically) in several publications. An early example is, e. g., a publication by R. R. Ernst and R. E. Morgan (1973).

The presented analysis is valid only for simple 90° excitations; FLASH sequences or steady-state pulse sequences which refocus the transverse magnetization as well will exhibit a different behavior.

Of course, optimizing T_R as described above is *not* an option if T_1-weighted or T_2-weighted images are to be acquired. In these cases, the optimal value of T_R depends on the desired contrast and not on the total SNR.

(This is a shortened version of a slightly longer pdf document that contains some more details and discusses also a few special cases.)

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